80% Exam

20% CA

↓

4 groups

Not >3 in a group

2 Presentation (10%)

Griffiths Intro to Electronics

4 ED (Jackson)

$17^{th} \to 24^{th}$ March

× $25 - 5^{th}$ April

Syllabus

- Review of MP204 – Coulomb's Law
  Gauss Law
  Faraday etc
- Scalar, Vector Potential
- Method of images
- multipole expansions
- $\vec{E}$, $\vec{B}$ for various charge configurations
- Magnetostatics $\vec{\nabla} \cdot B = 0$, $\vec{\nabla} \cdot E = \frac{\rho}{\varepsilon_0}$

- Dielectrics materials and polarization
- Relativistic formulation formulation
  (Jackson maybe used)
- Radiation and energy transport

Wed 12-1

Friday 9-10
12-1

$\vec{\nabla}\cdot\vec{B} = 0$ implies no magnetic monopole

$\vec{B} = 0$

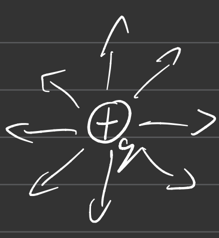
$\Rightarrow \vec{\nabla}\times\vec{E} = 0 \longrightarrow$

$-\vec{\nabla}V = \vec{E}$

$\vec{\nabla}\times(\vec{\nabla}V) = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x V & \partial_y V & \partial_z V \end{vmatrix}$$

$(\partial_x\partial_y - \partial_y\partial_x)V = 0$ etc



Dir $\hat{r}$

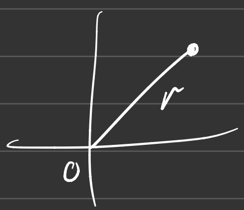
$$\vec{\nabla} \cdot \vec{r} = \partial_x x + \partial_y y + \partial_z z = 3$$

$$\vec{\nabla} r$$

$$\vec{r} = (x, y, z)$$

$$\vec{r} = (x_1, x_2, \ldots x_n) \Rightarrow \vec{\nabla} \cdot \vec{r} = n$$

$$\vec{r} = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$



$$\vec{\nabla} r = (\partial_x r)\hat{i} + (\partial_y r)\hat{j} + (\partial_z r)\hat{k}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \text{sim for } y, z$$

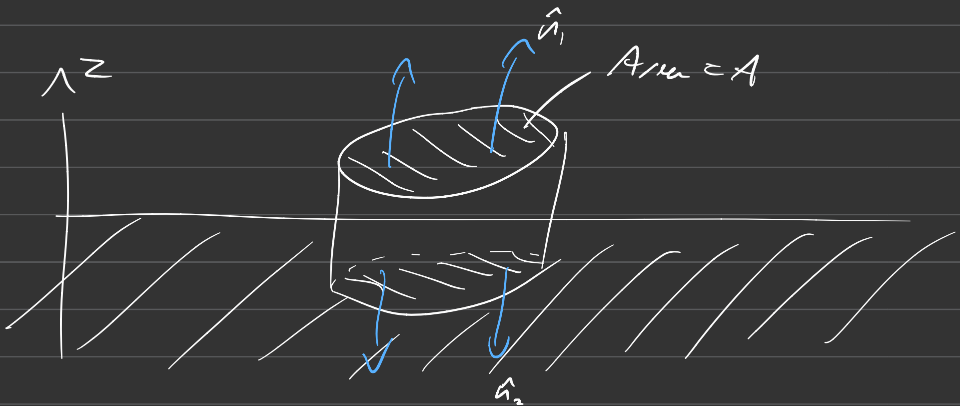
$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

Stokes Theorem

Gauss Law

$\Big\}$ helps with integrals



$$E \cdot 2A = \frac{\sigma A}{\varepsilon_0}$$

charge density σ

$$E = \frac{\sigma}{2\varepsilon_0}$$

Electrostatic

Magnetostatics (last 2)

Potential (11 problems)

$\downarrow$

image charge

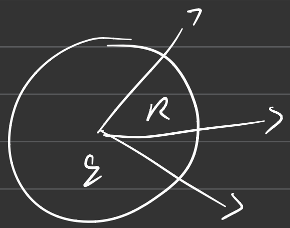
Various boundary conditions

$$\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

or $\nabla^2 V = 0$

Gauss' Law

Compute Electric flux



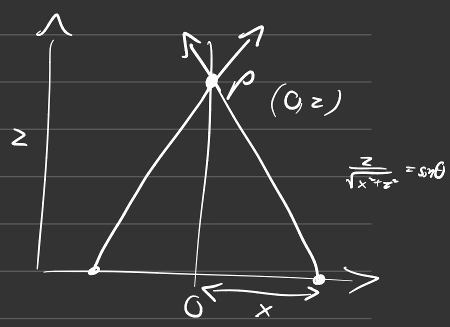
$$|E| \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$|E| = \frac{q}{4\pi\varepsilon_0 R^2}$$

↳ Coulomb's Law

Pg 62 Example 2.1

$$\vec{E} = E\hat{z}\int_0^L \frac{2\lambda z\,dx}{(z^2+x^2)^{3/2}}$$



$$2\lambda z \frac{x}{(x^2+z^2)^{1/2} z^2}\Big|_0^L$$

λ = charge density

$$= \frac{2\lambda L}{(L^2+z^2)^{1/2}}\Big|$$

Divergence Theorem

$$\int \vec{\nabla} \cdot \vec{E} \, dv = \oint \vec{E} \cdot d\vec{a}$$

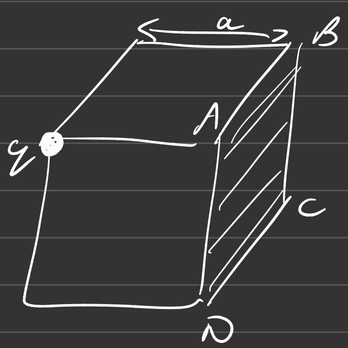
$$= \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \int \rho \, dv$$

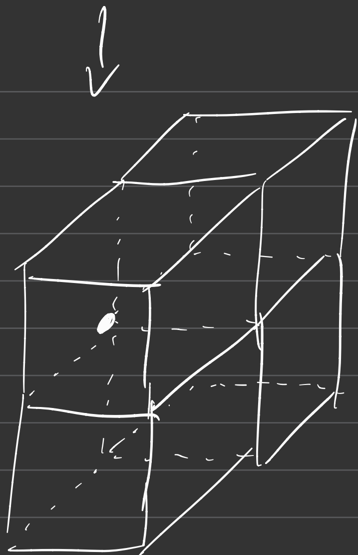
Maxwells Equation

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$

Example (Pg 70 Prob 2.10)



Calculate the flux from the face ABCD

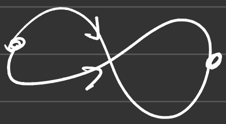


8 cubes starts on top and sides

$$\frac{q}{24\varepsilon_0}$$

Radius of enveloping sphere

$= a\sqrt{3}$



Such field lines don't exist except at charges

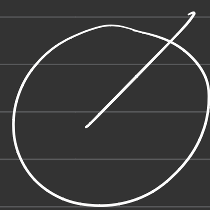
$$v = \frac{\hat{r}}{r^2}$$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r^2}{r^2}\right) = 0$$

non zero iff $r = 0$

$$\oint \vec{v} \cdot d\vec{a} = \int \frac{R^2 \sin\theta \, d\theta \, d\phi}{R^2}$$

$$= 4\pi$$



$$\nabla \cdot \left(\frac{\vec{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

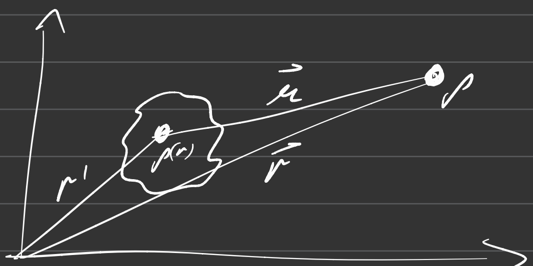
$$\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$$

Can $\vec{F}$ be a magnetic or electric field

$$\vec{\nabla} \cdot \vec{F} = 3 \qquad \vec{\nabla} \times \vec{F} = 0$$

Cannot be a magnetic field
Can be an electric field

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,\hat{r}\,dv}{r^2}$$



Method of Image (Pg 124)

We don't know ρ most of the time.
So how to calculate $\vec{E}$?

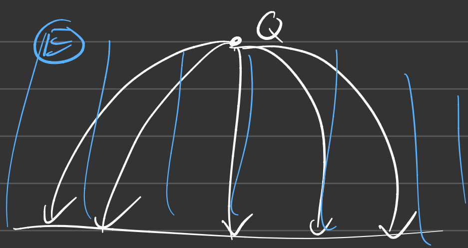


Image can't be in the region

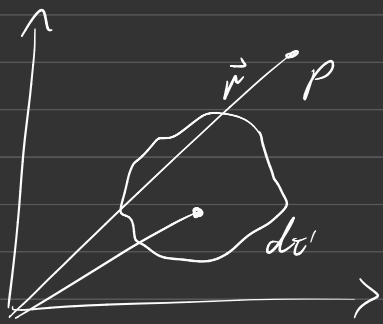
$V = 0$

grounded conductor

Potential

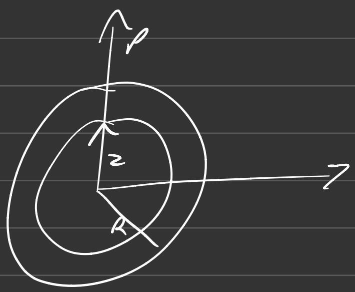
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}} \qquad \mathscr{r} = |\vec{r} - \vec{r}\,'|$$

Amount of work done to bring in a unit ~~time~~ charge to $\vec{r}$

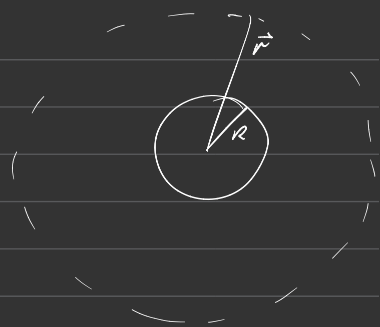


$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

$V(\vec{r})$ due to a spherical shell of uniform charge density ρ



$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} = \frac{4\pi R^2 \rho}{\varepsilon_0}$$



$$E = \frac{R^2 \rho}{\varepsilon_0 r^2}$$

$V(z)$ use the formula
and find $V(z)$

- Point charge $q_1, q_2, \ldots, q_n$

Q What is the energy for making this configuration

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{i<j} \frac{q_i q_j}{|r_i - r_j|}$$

$$= \sum_{i=1}^{n} \frac{q_i}{8\pi\varepsilon_0} \sum_{i\neq j} \frac{q_j}{|r_i - r_j|}$$

$$= \frac{1}{2} q_i V_i$$

We can generalize for continuous charge distributions

$$W = \frac{1}{2}\int \rho V \, d\tau$$

$$= \frac{1}{2}\int_{\text{all space}} \varepsilon_0 E^2 \, d\tau$$

How ~~to~~ show use Gauss law, differential form, use integration by parts

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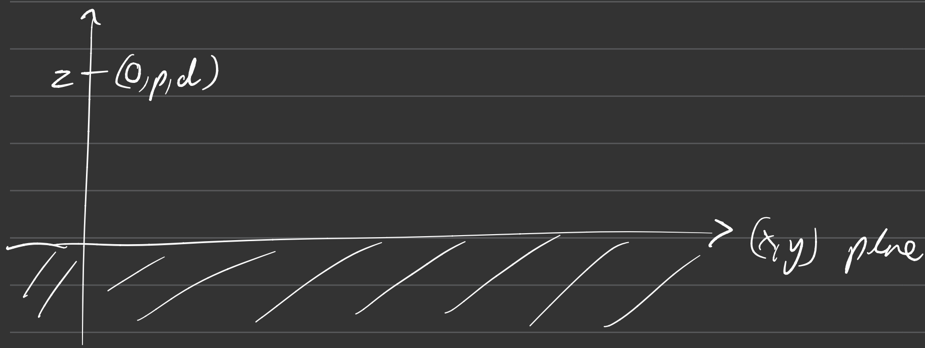


Grounded conducting plane

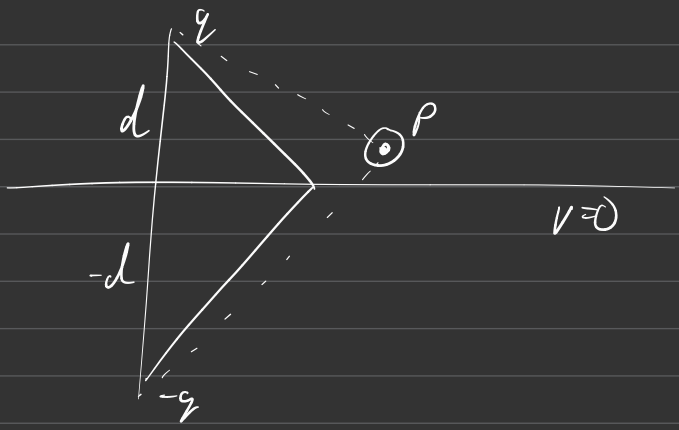
$V = 0$ on (x-y plane)

Uniqueness Theorem

V satisfies Laplaces equation away from q upto x,y plane

$$\nabla^2 V = 0$$

Laplaces Equation satisfied at the boundary including ∞, then



$$V_P = \frac{q}{4\pi\varepsilon_0 (x^2+y^2+(z-d)^2)^{\frac{1}{2}}} - \frac{q}{4\pi\varepsilon_0 (x^2+y^2+(z+d)^2)^{\frac{1}{2}}}$$

$$E = -\vec{\nabla} V$$

Can you calculate the surface charge on the conductor

Another formula

$$\left.\frac{\partial V}{\partial n}\right|_{surface} = -\frac{\sigma}{\varepsilon_0}$$

n is ~~the~~ normal

To be asking

$$\left.\frac{\partial V}{\partial z}\right|_{surface} = -\frac{\sigma}{\varepsilon_0}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-(z-d)}{\mathcal{r}^3} + \frac{(z+d)}{\mathcal{r}^3}\right]$$

$$\mathcal{r} = \left(x^2+y^2+(z-d)^2\right)^{\frac{1}{2}}$$

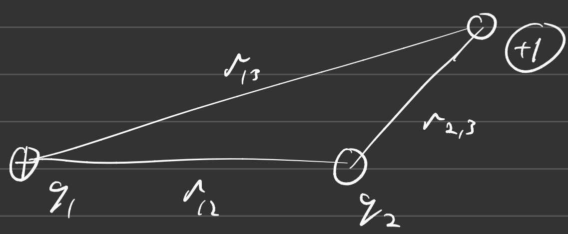
$$-\frac{\sigma}{\varepsilon_0} = \frac{-2qd}{4\pi\varepsilon_0 \mathcal{r}^3}$$

Total charge $Q = \int \sigma \, da$

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd\, r\, dr\, d\theta}{2\pi(r^2+d^2)^{\frac{3}{2}}}$$

$$r = (x^2 + y^2)^{\frac{1}{2}}$$

$$Q = \frac{qd}{\sqrt{r + d^2}} \Big|_0^\infty = -q$$



$$W(r_{12}) = \frac{q_1 q_2}{r_{12}}$$

$$W(r) = \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}$$

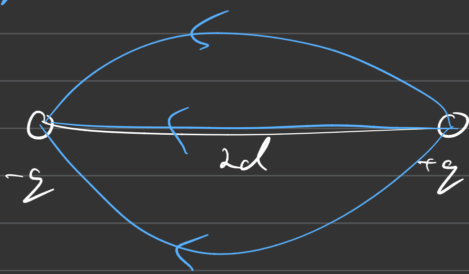
Work done in an electrostatic system
q_i changes at r_i points

$$\sum_{i=1}^{n} q_i V(r_i)$$

For continuous systems $W = \frac{1}{2} \int \rho V(r') d\tau$

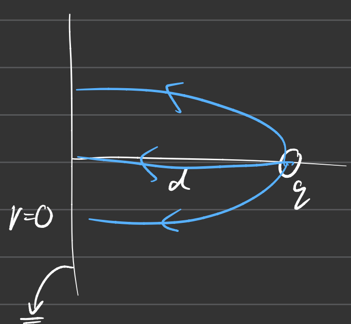
$$W = \frac{1}{2}\varepsilon_0 \int E^2 \, d\tau$$

Example



$$W = \frac{-q^2}{4\pi\varepsilon_0(2d)}$$

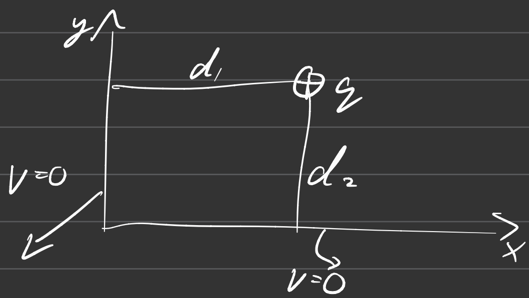
Example



conductor case

$$W = \frac{1}{2}\frac{(-q^2)}{4\pi\varepsilon_0 2d}$$

Example

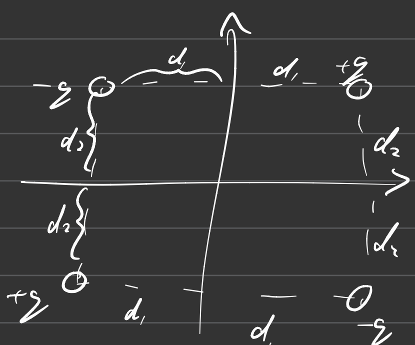


Same infinite conductors

(x, z) and (y, z) plane

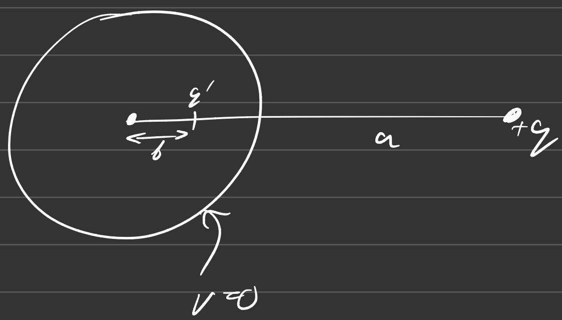
in $x \geq 0$, $y \geq 0$, $z \geq 0$

$V(x, 0, z)$, $V(0, y, z)$



$$V = \frac{q}{d_2^2 + z^2 + x^2} - \frac{q}{d_2^2 + z^2 + x_1^2} + V(q_3) + V(q_4)$$

Example



grounded Conducting sphere R radius

V = 0 on the surface of the sphere

$$V(P_1) = \frac{q}{a-R} + \frac{q'}{R-b} = 0 \quad - (1)$$

$$V(P_2) = \frac{q}{R+a} + \frac{q'}{R+b} = 0 \quad - (2)$$

$$q' = -\frac{q(R-b)}{a-R}$$

$$q' = \frac{-q(R+b)}{(R+a)}$$

$$(R-b)(R+a) = (R+b)(R-a)$$

$$= R^2 - ab + bR - aR$$

$$=$$

$$R^2 = ab$$

$$b = \frac{R^2}{a}$$

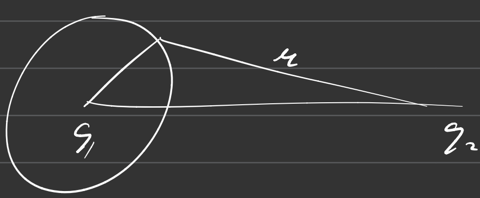
$$q' = \frac{-q\left(R - R^2/a\right)}{a-R}$$

$$= \frac{-qR}{a}$$

If $\quad q' = \frac{-Rq}{a}$

$$b = \frac{R^2}{a}$$

$V(R, \theta, \phi)$



$$V_q(P) + V_{q'}(P)$$

$$= \frac{q}{r_1} - \frac{qR}{a r_2}$$

$$= \frac{q}{\sqrt{R^2 + a^2 - 2Ra\cos\gamma}} - \frac{qR}{a\sqrt{R^2 + \frac{R^4}{a^2} - \frac{2R^3}{a}\cos\gamma}}$$

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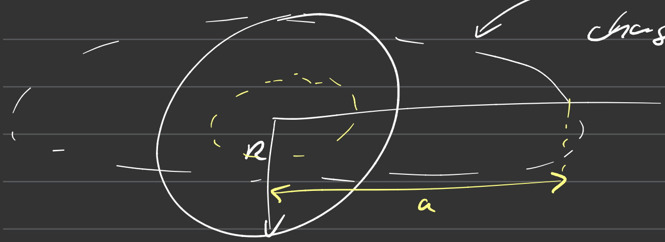
circular ring of charge density $\rho(\theta')$

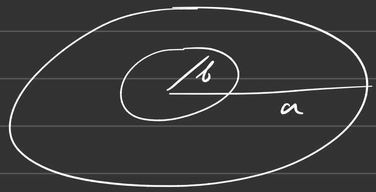


Sphere of radius R, $+q$ at distance 'a' from center

$$q' = -\frac{R}{a} q$$

placed at $\frac{R^2}{a}$ from center

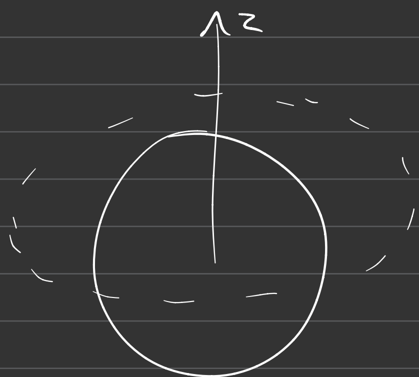
Image problem this rings



$$b = \frac{R^2}{a}$$

$$\rho' \sim \rho \frac{R}{a}$$

Dalan → Green's functions



$$V(z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{z^2+a^2}} - \frac{R}{a\sqrt{z^2+\frac{R^4}{a^2}}}\right]$$

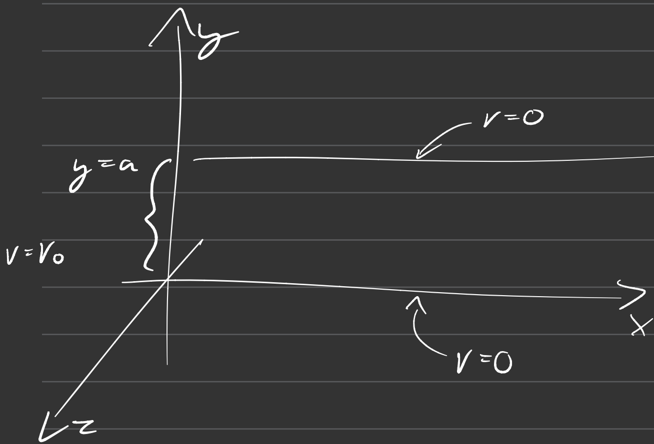
Various config possible

Solutions not always possible by images

$$\nabla^2 V = 0$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Example



$$V(x, 0, z) = 0$$

$$V(x, a, z) = 0$$

$$V(0, y, z) = V_0$$

$$\nabla^2 V = 0 \quad \text{Laplace}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V = X(x)\,Y(y)$$

$$\frac{1}{X}\frac{d^2V}{dx^2} = -\frac{1}{Y}\frac{d^2V}{dy^2} = k^2$$

$$X = Ae^{kx} + Be^{-ky}$$

$$Y = C\sin(ky) + D\cos(ky)$$

$$x \longrightarrow \infty \qquad V = 0$$

$$X = Be^{-kx}$$

$$Y = \sin(ky)$$

where $k = \dfrac{n\pi}{a}$

$$V = \sum_n c_n \sin\left(\frac{n\pi}{a}y\right) e^{-\frac{\pi n x}{a}}$$

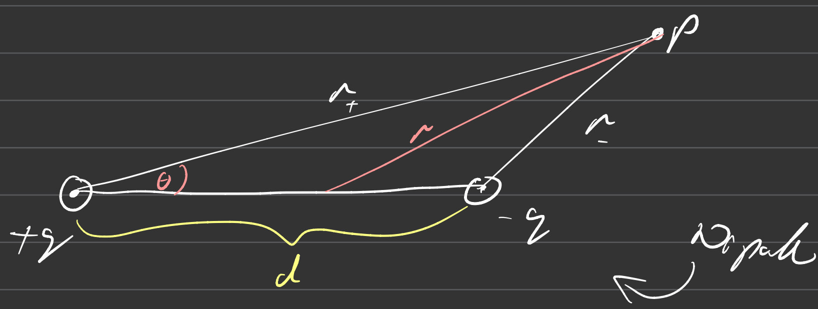
⋮

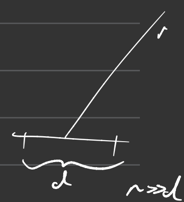
Look at
griffiths

⋮

$$V = \frac{4V_0}{\pi}\sum \frac{e^{-\frac{n\pi x}{a}}}{n}\sin\left(\frac{n\pi x}{a}\right)$$

Baby step towards multipole expansions





$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left(r^2 + \frac{d^2}{4} - 2rd\cos\theta\right)^{\frac{1}{2}}} - \frac{1}{\left(r^2 + \left(\frac{d}{2}\right)^2 + 2rd\cos\theta\right)^{\frac{1}{2}}}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1 + \frac{d}{r}\cos\theta\right) - \frac{1}{r}\left(1 - \frac{d}{r}\cos\theta\right)\right]$$

$$V(r) \sim \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} = \frac{q}{4\pi\varepsilon_0 r^2}d\cos\theta$$

$-q$ $+q$ Quadrupole

$+q$ $-q$

Laplace equation in Cartesian coordinate

$V \sim \sin ky \, e^{-kx}$

Laplace Equation in Spherical Coordinates

$\nabla^2 V = 0 \qquad V = V(r, \theta)$

$= R(r)\,T(\theta)$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{=l(l+1)} = \frac{-1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

$R = r^l$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(l r^{l+1}\right) = l(l+1)r^l$$

$$R \sim r^{-(l+1)}$$

$$r^2 \frac{dR}{dr} = -(l+1) r^{-l}$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -(l+1) \frac{d}{dr} r^{-l}$$

$$= l(l+1) r^{-(l+1)}$$

$$R = A_l r^l + \frac{B_l}{r^{l+1}}$$

and l is an integer

$$\frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right) = -l(l+1) \sin\theta \, T$$

$$T = P_l(\cos\theta)$$

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

for $l = 0$ $\quad P_0(x) = 1$

$$P_1(x) = \cos\theta$$

$$\nabla^2 V(r,\theta) = 0$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Properties of $P_l(x)$

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \delta_{l,l'} \frac{2}{2l+1}$$

$$\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta$$

$$= \frac{2}{2l+1} \delta_{l,l'}$$

$A_l \sim \sum \int P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta$

$$V(r=R,\theta) = V_0(\theta)$$

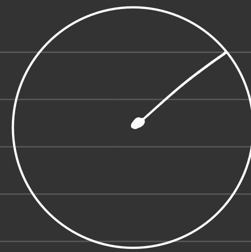
$$V(r,\theta) = \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta)$$

Using orthogonality conditions A_l B_l

$V(r,\theta)$ is a continuous function at R

and $V(r \to \infty, \theta) = 0$

$V(r \to 0, \theta)$ also finite



$$V_0(\theta) = K\sin^2\left(\tfrac{\theta}{2}\right)$$

$$= K\tfrac{1}{2}(1-\cos\theta)$$

$$V_{in}(r,\theta) = \sum_{l=0}^{1} A_l r^l P_l(\cos\theta)$$

inside

at $r = R$

$$V_{in} = V_{out}$$

$$V_{out}(r,\theta) = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$A_l P_0(\cos\theta) + A_1 R P_1(\cos\theta)$$

$$= \frac{B_0}{R} P_0(\cos\theta) + \frac{B_1}{R^2} P_1(\cos\theta)$$

$$A_0 = \frac{B_0}{R}$$

$$A_1 = \frac{B_1}{R^3}$$

$$V_{in} = A_0 + A_1 r$$

$$V_{in} = \frac{K}{2} - \frac{K r}{2R}$$

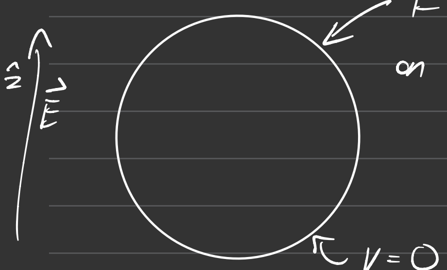
$$V_{out} = \frac{K}{2}\frac{R}{r} - \frac{K}{2R}\frac{R^2}{r^2}$$

$$= \frac{K}{2}\left(\frac{R}{r} - \frac{R^2}{r^2}\right)$$

$$V_0(r=R,\theta) \sim (\cos n\theta) P_n(\cos\theta)$$

Example

$E = 0$ (discontinuous at R) on the surface



conductor (spherical) placed in uniform electric field

$$E(r\to\infty) = E\hat{z}$$

$$V(r\to\infty) = -Ez = -Er\cos\theta$$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l P_l(\cos\theta) R^l + \frac{B_l}{R^{l+1}} P_l(\cos\theta) = 0$$

$$\frac{\partial}{\partial r}\left(V(r\to\infty, \theta)\right) = -E$$

$r \gg R$

$$V(r\to\infty, \theta) = -Er\cos\theta$$

$$= A_1 \text{ term}$$

$$A_1 = -E$$

Last class we used spherical coordinates for

$\nabla^2 V = 0$, $V = V(r, \theta) = R(r)\,T(\theta)$

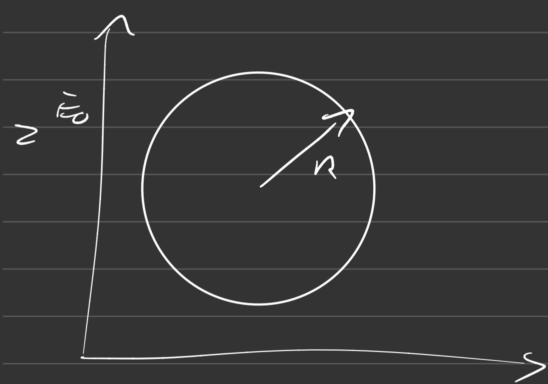
so Azimuthal symmetry is present

$$\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = l(l+1) \qquad (A)$$

$\cos\theta = x$

and rewrite (A)

wrt x



$V(R, \theta) = 0$

$E \neq 0$ at infinity. So we have a term in ~~the~~ potential outside ~~the~~ sphere also

$$V(R,\theta) = 0 = A_l R^l + \frac{B_l}{R^{l+1}}$$

$$\Rightarrow B_l = -A_l R^{2l+1}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta)$$

$$A_1 = -E_0$$

and rest A_l would be 0

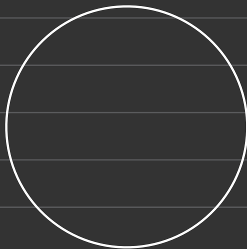
$$B_1 = \frac{R^3}{r^2}$$

$$V(r,\theta) = E_0\left(\frac{R^3}{r^2} - r\right)\cos\theta$$

$$\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} = \sigma$$

$$\varepsilon_0 E_0 \left(-\frac{2R^3}{R^3} - 1\right)\cos\theta$$

$$= 3\varepsilon_0 E_0 \cos\theta$$



$\sigma = k\cos\theta$ at ~~the~~ surface

$$\left(A_l R^l + \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta) = k\cos\theta$$

$$B_1 = A_1 R^3$$

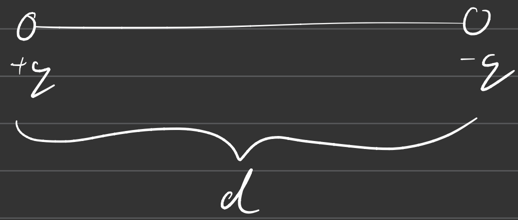
$$-l A_l R^{l-1} - \frac{B_l (l+1)}{R^{l+2}}$$

$$= -\frac{k}{\varepsilon_0}$$ for $l = 1$

Multipole Expansion

$$V_{dipole}(r, \theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

where $p = qd$



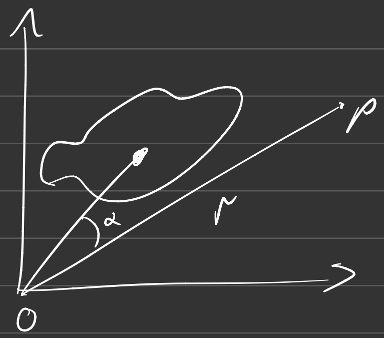
$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^2}$$

$$E_\phi = 0$$

$$E_{dipole} \sim \frac{1}{r^3} \qquad V_{dipole} \sim \frac{1}{r^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d\tau'$$



$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\alpha}$$

$$= r\sqrt{1 + \underbrace{\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha}}$$

$$= r\sqrt{1 + \varepsilon}$$

small compared to $|r|$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r}(1+\varepsilon)^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 \dots\right)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$-\frac{1}{2}\frac{\varepsilon r^2}{r}$$

$$\varepsilon = -2\frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2$$

So the term $\frac{r'}{r^2}$ $\cos\alpha = P_1 \cos$

<u>Quadrupole</u>

$$\frac{(r')^2}{r^3}$$

$$\frac{(r')^2}{r^3} : -\frac{1}{2} + 4\cos^2\alpha\,\frac{3}{8}$$

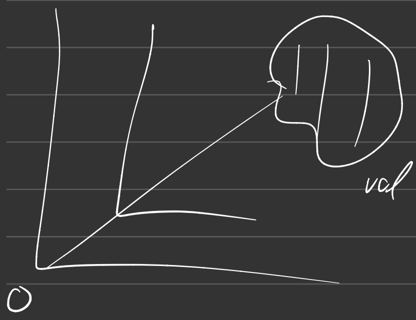
$$= \frac{3}{2}\cos^2\alpha - \frac{1}{2}$$

$$= P_2(\cos\alpha)$$

Dipoles

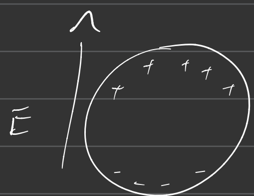
$$\vec{P}_1 = \int_{vol} \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau' = \int \rho(\vec{r}\,'')\,\vec{r}\,''\,d\tau'$$

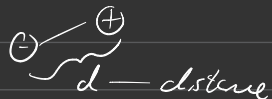
$$= \int (\vec{r}\,' + \vec{a})\,\rho(\vec{r}\,')\,d\tau'$$

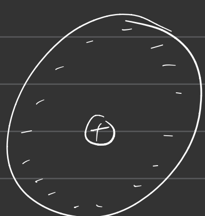


$$= \int r''\rho(r'') + a\underbrace{\int \rho(\vec{r}\,'')\,d\tau}_{\text{if } = 0}$$

Objects which can get polarised under an external electric field

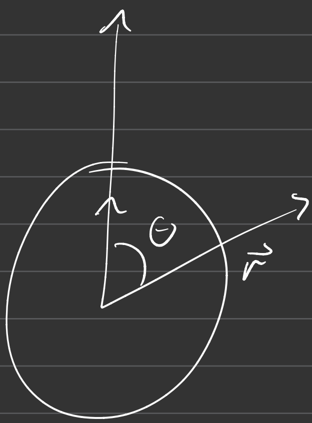






Dipole moment

$$V_{dip} = \frac{\hat{r} \cdot p\cos\theta}{4\pi\varepsilon_0 r^2}$$



$$E_r = -\frac{\partial V}{\partial r}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

E_{dipole} can be given by $\hat{r}$ and $\vec{p}$ without any notion of coordinate frame

$$E_{dip} = E_r\hat{r} + E_\theta\hat{\theta}$$

$$V(\vec{r}') = V_{dip} = \frac{1}{4\pi\varepsilon_0}\int \rho(r')\, r \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$V_{dip} = \frac{1}{4\pi\varepsilon_0}\int \vec{P}(r)\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \vec{P}\cdot\nabla\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{\int \nabla\left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau}_{\text{surface integral}} - \underbrace{\int \nabla'\vec{P}\cdot\frac{1}{\mathscr{r}}\, d\tau'}_{\text{Volume integral}}\Bigg]$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}}{\mathscr{r}}\, d\vec{a} - \int (\nabla\cdot\vec{P})\frac{1}{\mathscr{r}}\, d\tau'$$

$$\sigma_b = \vec{P}\cdot\hat{n}, \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$

Bound Surface Charge density

Bound Charge density

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_{free} + \rho_b}{\varepsilon_0}$$

$$\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_{free} + \rho_b$$

$$= \rho_{free} - \vec{\nabla} \cdot \vec{P}$$

$$\nabla (\varepsilon_0 \vec{E} + \vec{P}) = \rho_{free}$$

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_{free}}$$

Gauss law in dielectrics

Example

$$\int \vec{\nabla} \cdot \vec{D} = Q_{free}$$

$$L\lambda = Q_{free}$$

$$2\pi s L D = Q_{free} = \lambda L$$

$$\vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

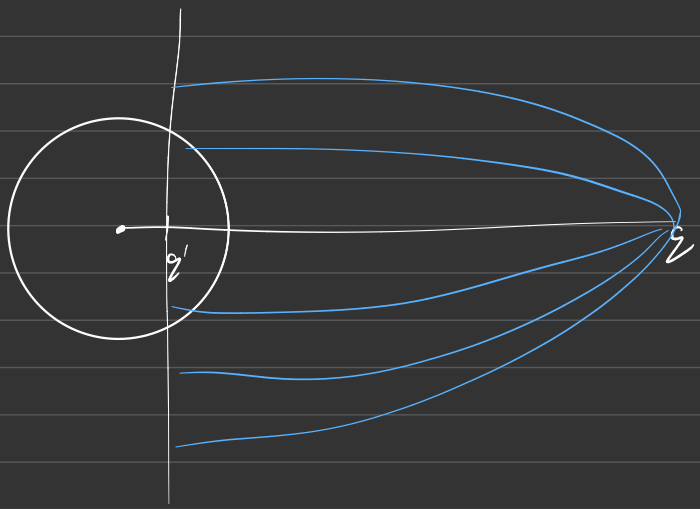
and outside $\vec{P} = 0$

$$\vec{E} = \frac{D}{\varepsilon_0}$$

$$= \frac{\lambda}{2\pi s \varepsilon_0}\hat{s} \qquad s > a$$

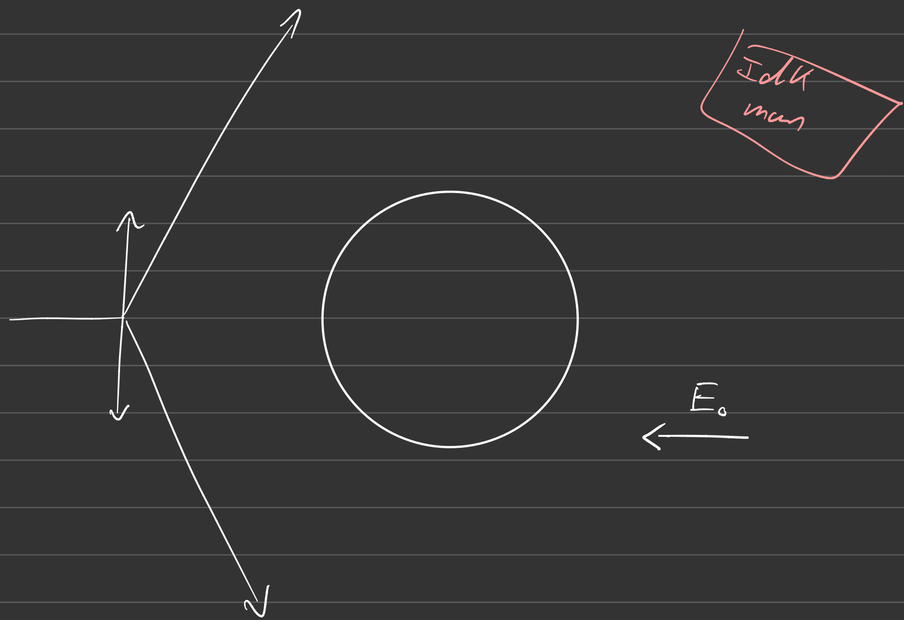
$$W_E = \frac{\varepsilon_0}{2} \int_{\substack{\text{all}\\\text{space}}} E^2 \, d\tau$$

$$W_{mag} = \frac{1}{2\mu_0} \int_{\substack{\text{all}\\\text{space}}} B^2 \, d\tau$$



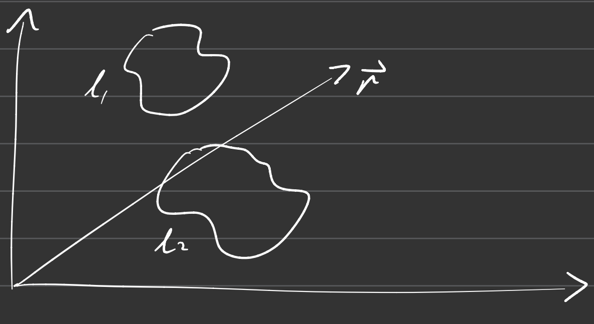
Capacitance $\quad C = \frac{Q}{V}$

Q11 Potentials



Q10 ~~Potentials~~

Green's reciprocity theorem



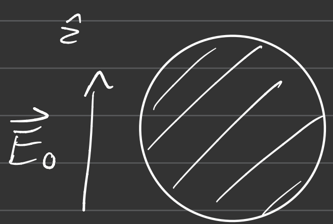
$V_1(\vec{r})$

$V_2(\vec{r})$

Then $\displaystyle\int_{\text{all space}} \rho_1 V_2 \, d\tau = \int_{\text{all space}} \rho_2 V_1 \, d\tau$

$E_1 \cdot E_2$

$\int \nabla V_1 \cdot E_2 = \int E_1 \cdot \nabla V_2$



Sphere of radius R is a dielectric medium of ε_r

$\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0} = H\lambda_e$

linear dielectrics

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\downarrow$$

$$E, P$$

$V_{in} = V_{out}$ at $r = R$

$$\varepsilon \frac{\partial V}{\partial r}\Big|_{r=R} = \varepsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

inside outside

$$D_{above} = D_{below}$$

$$V_{in} = \sum_l A_l P_l (\cos\theta)\, r^l$$

$$V_{out} = \sum_l B_l \, r^{-(l+1)} P_l(\cos\theta$$

$P_l(x)$ and $P_{l'}(x)$ are orthogonal

$$A_\ell = B_\ell = 0 \quad \text{iff} \quad \ell \neq 1$$

$$A_1 R = \frac{B_1}{R^2} - E_0 R$$

$$\varepsilon A_1 = -\varepsilon_0 \left(E_0 + \frac{B}{R^3} \right)$$

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_r A_1 = -\left(E_0 + \frac{2B}{R^3} \right)$$

$$\frac{\varepsilon_r A_1}{2} = -\left(\frac{E_0}{2} + \frac{B}{R^3} \right)$$

$$A_1 + \frac{\varepsilon_r A_1}{2} = -\frac{3}{2} E_0$$

$$A_1 (2 + \varepsilon_r) = -3 E_0$$

$$A_1 = -\frac{3}{2 + \varepsilon_r} E_0$$

$$\frac{B_1}{R^3} = A_1 + \varepsilon_0$$

$$= E\left(1 - \frac{3}{2+\varepsilon_r}\right)$$

$$= E_0\left(\frac{\varepsilon_r - 1}{2+\varepsilon_0}\right)$$

$$B_1 = E_0 R^3\left(\frac{\varepsilon_r - 1}{2+\varepsilon_0}\right)$$

$$\frac{\partial V}{\partial n} - \frac{\partial V}{\partial n} = \frac{-\sigma}{\varepsilon_0}$$

$Q_{enc} = \sigma A$, σ = charge density

$$E_{above} \cdot A + E_{below} A + E_{sides}^{\to \varepsilon} = \frac{\sigma A}{\varepsilon_0}$$

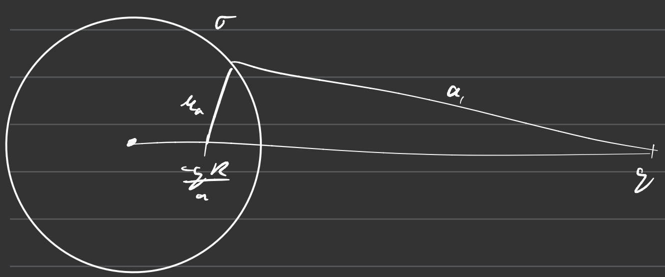
$$\left(E_{above} - E_{below}\right) = \frac{\sigma}{\varepsilon_0}$$

$$\hat{n} \cdot \vec{E} = -\frac{\partial V}{\partial n}$$

So ~~this~~ gives

$$\left.\frac{\partial V}{\partial n}\right|_{out} - \left.\frac{\partial V}{\partial n}\right|_{in} = \frac{\sigma}{\varepsilon_0}$$

Conducting sphere positive charge on outside



This ~~the~~ one Sean did ; think

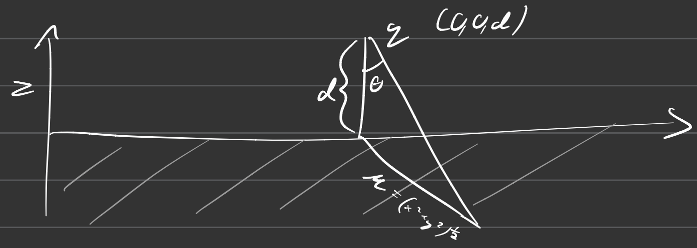
$$\left.\frac{\partial V}{\partial r}\right|_{r=R}$$ only one term present

$$V = \frac{q}{r_1} - \frac{qR}{a\, r_2}$$

"This will be on ~~the~~ exam, you can assume that"

$$\int \sigma\, da = q_{induced}$$

"Paul will give lecture on greens functions St Patrick's day week"



$z > 0$ filled with dielectric of ε_r

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{P} = 0$$

σ_b on surface

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$$= P_z = \varepsilon_0 \chi_e E_z \Big|_{\text{on } xy \text{ plane}}$$

(1) $E_z^{(1)} = \frac{-q\cos\theta}{4\pi\varepsilon_0(r^2+d^2)}$

$$\cos\theta = \frac{d}{(r^2+d^2)^{\frac{1}{2}}}$$

$$E_z^{(1)} = \frac{-qd}{(r^2+d^2)^{\frac{3}{2}}} \quad \text{at} \quad z=0$$

$$\frac{\sigma_b}{2\varepsilon_0}\hat{n} = E_z^{(2)}$$

$$E_z = E_z^{(1)} + E_z^{(2)}$$

$$\sigma_b = \varepsilon_0 \chi_e (E_z^{(1)} + E_z^{(2)})$$

$$= \varepsilon_0 \chi_e \left(\frac{-qd}{(r^2+d^2)^{\frac{3}{2}}} \frac{1}{4\pi\varepsilon_0} + \frac{\sigma_b}{2\varepsilon_0} \right)$$

$$\sigma_b = \left(1+\frac{\chi_e}{2}\right) = -\frac{qd\chi_e}{4\pi(r^2+d^2)^{\frac{3}{2}}}$$

$$q_{induced} = \int \sigma_b \, 2\pi r \, dr$$

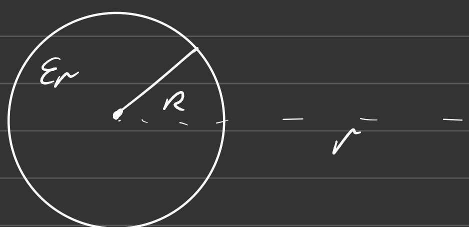
$$= \int_0^\infty \frac{r}{(r^2+d^2)^{\frac{3}{2}}} \, dr$$

$$= \int_0^\infty \frac{w\,dw}{w^3} \qquad w^2 = r^2 + d^2$$
$$2w\,dw = 2r\,dr$$

$$\int \frac{\sin\theta \, d\theta}{\sqrt{r^2+d^2+dr\cos\theta}}$$

Energy for a dielectric system

$$W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}$$



ρ_f = free charge density

$$Q_{total} = \int \rho_f \, 4\pi r^2 dr$$

$=$ $\qquad r < R$

To find $\vec{D}$

$\underline{r \leq R}$

$$4\pi r^2 D = \frac{4}{3}\pi r^3 \rho_f$$

$$\Rightarrow \quad \vec{D} = \frac{\rho_f}{3}\vec{r}$$

$\underline{r > R}$

$$4\pi r^2 D = \frac{4}{3}\pi R^3 \rho_f$$

$$\vec{D} = \frac{R^3}{3r^2}\rho_f \hat{r}$$

$$E(\vec{r}) = \frac{\vec{D}}{\varepsilon_0}$$

$$\vec{E}(\vec{r}) = \begin{cases} \frac{\rho_f \vec{r}}{3\varepsilon_0} & \text{if} \quad r \le R \\ \frac{\rho_f R^3}{3\varepsilon_0 r^2} & \text{if} \quad r > R \end{cases}$$

$$W = \frac{\varepsilon_0}{2}\int_R^\infty E^2 \, d\tau + \frac{1}{2}\int_0^R \vec{D}\cdot\vec{E} \, d\tau$$

$$= \boxed{\frac{\varepsilon_0}{2} \frac{\rho_f^2}{9\varepsilon_0^2}} \int^R 4\pi r^2 \, r^2 dr$$

without dielectric

$$\frac{\varepsilon_0}{2}\int E^2 \, d\tau$$

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\text{Diff} = \frac{2\pi}{45\varepsilon_0} \rho_f^2 \frac{R^5}{\varepsilon_r^2} (\varepsilon_0 - 1)$$

Magnetostatics

(1) Part of Review of MP204

(2) Parallels with electrostatics

(3) Vector potentials

A) Vector Potentials are <u>not</u> unique

Gauge transformations

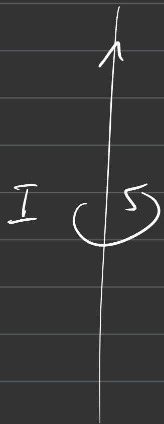
This leads to electrodynamics and relativistic formulation

B) Retarded and advanced potentials

$$V(\vec{r}) = \# \int \frac{\rho(r', t)\, d\tau'}{\mathscr{r}}$$

electromagnetic waves travel with velocity of light

Example



Magnetic field

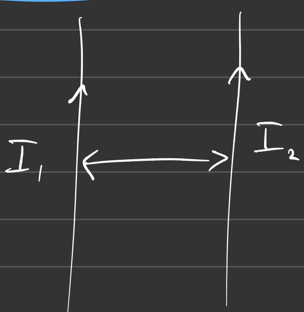
$$\vec{B}$$

$$\vec{B} = \mu_0 \frac{I_{enclosed}}{2\pi s} \hat{\phi}$$

Ampere's Law says

$$B \cdot 2\pi s = \mu_0 I_{enclosed}$$

Example



Force on one of them per unit length

$$F_{mag} = q(\vec{v} \times \vec{B})$$

$$\vec{B} = \frac{\mu_0 I_1 \hat{\ell}}{2\pi d}$$

$$qv = I_2\, dl$$

$$\text{Force}/\text{unit length} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Ampere's Law in differential form

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

No magnetic monopole exist

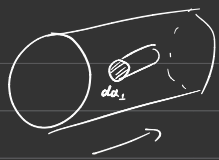
$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Div $= 0$ for vector V can be expressed as

$$\vec{\nabla} \times \vec{A}$$

↳ magnetic vector potential

$\vec{J}$: Current density over a volume



$\vec{K}$: Surface Current density

$I = J\, da_{\perp}$

$\vec{I}$: line Current

Steady Current

$$\vec{\nabla} \cdot \vec{J} = 0$$

on $\dfrac{\partial \rho}{\partial t} = 0$

$$I = \int |J|\, da_{\perp} = \oint \vec{J} \cdot d\vec{a}$$

$$= \int \frac{\partial \rho}{\partial t}\, d\tau = \int \vec{\nabla} \cdot \vec{J}\, d\tau$$

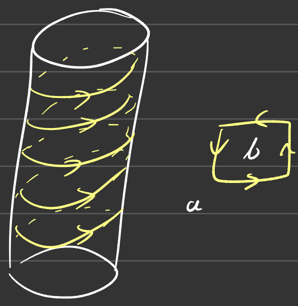
Continuity Equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

and steady Current $\Rightarrow$ $\left.\begin{array}{l} \vec{\nabla} \cdot \vec{J} = 0 \\ \dfrac{\partial \rho}{\partial t} = 0 \end{array}\right\}$ Ampere's Law holds with steady current only

Solenoid

Solenoid infinitely long



$B(a) - B(b) = 0$

or $B(a) = B(b)$

$B = 0$ at ∞

$B = 0$ everywhere outside solenoid

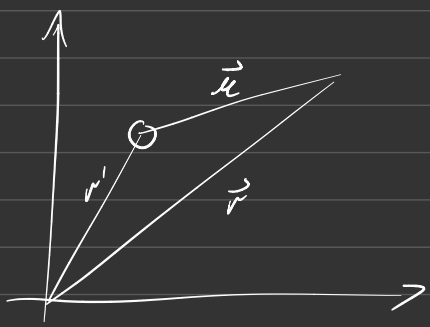
$$\oint B\, dl = BL = \mu_0 n I L$$

$$B = \mu_0 n I$$

If $n = n_0$ of loops per unit length

Biot Savart Law

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$



$\vec{\nabla} \cdot \vec{J} = 0$ steady current

$\vec{\nabla} \cdot \vec{B} = 0$ to show

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \cdot \left(\vec{J}(r') \times \frac{\hat{r}}{\mathscr{r}^2} \right) d\tau'$$

Now use some vector calculus results

$$\nabla \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} (\vec{\nabla} \times \vec{J}) - \vec{J} \cdot \left(\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

$$\nabla \times J = 0 \implies \vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$$

$\vec{\nabla}$ in cartesian co-ordinates or spherical coordinates

Biot Savart -> Ampere's Law

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{J\times\hat{r}}{r^2}\right)d\tau'$$

$$\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{r}}{r^2}\right) = J\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) - \left(\vec{J}\cdot\vec{\nabla}\right)\frac{\hat{r}}{r^2}$$

$$\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int 4\pi\delta^3(\vec{r})\,\vec{J}\,d\tau'$$

$$= \mu_0\vec{J}$$

$$\int\left(J\cdot\vec{\nabla}'\right)\frac{\hat{r}}{r^2}\,d\tau' = 0$$

$$= \int \nabla'\left(\vec{J}\cdot\frac{\hat{r}}{r^2}\right)d\tau' - \int\frac{\hat{r}}{r^2}\overset{\to 0}{\cancel{\left(\vec{\nabla}'\cdot\vec{J}\right)}}\,d\tau$$

$\vec{\nabla} \cdot \vec{J} = 0$ for steady currents

$$\int \nabla \cdot \left(\vec{J}(r') \cdot \frac{\hat{\imath}}{\imath^2} \right) d\tau'$$

$$= \oint \vec{J}(r') \cdot \frac{\hat{\imath}}{\imath^2} \, d\tau'$$

$= 0$ for localised charge dist

$J(r') = 0$ at $r' \to \infty$. Also if $J(r')$ is constant as $J(r')$

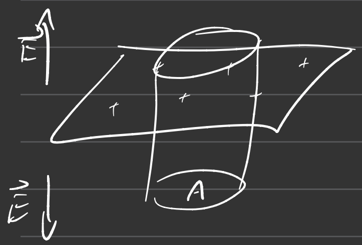
Still we have

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Ampere's law in different form

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enclosed}$$

$$= \oint (\vec{\nabla} \times \vec{B}) \, \vec{da} = \mu_0 \int \vec{J} \cdot \vec{da}$$



$$\frac{\sigma A}{\varepsilon_0} = \frac{Q_{enc}}{\varepsilon_0} = \oiint \vec{E}\, \vec{da}$$

$$= \oiint_{Bot} \vec{E}\, \vec{da} + \oiint_{side} \vec{E}\, \vec{da} + \oiint_{Top} \vec{E}\, \vec{da}$$

$$= -|E_{BOT}|\, A + |E_{top}|\, A$$

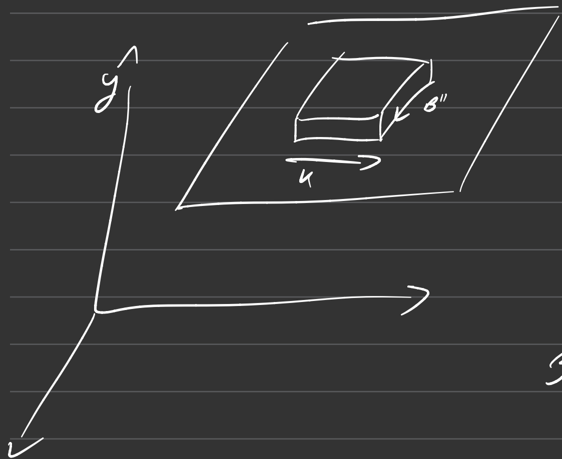
$$\Rightarrow E^{\perp}_{Top} - E^{\perp}_{Bot} = \frac{\sigma}{\varepsilon_0}$$

How to relate surface currents and $\vec{B}$ discontinuity

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \underbrace{\vec{B} \cdot \vec{da}} = 0$$

$$B^{\perp}_{top} - B = 0$$



$$\vec{B} = \# \int \frac{d\vec{l} \times \vec{r}}{r^2}$$

or $$\int \frac{\sigma \vec{K} \times \hat{r}}{r^2}$$

B along K $= 0$

$$B_x = 0$$

$$B_z^{up} = B_z^{down}$$

$$B_y(I) \neq B_y(I)$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\boxed{\vec{B}^{above} - \vec{B}^{below} = \vec{K} \times \hat{n}}$$

Green's Functions

MP361

$$Ly = f(x)$$

$$L = a_2(x)D^2 + a_1(x)D + a_0(x)$$

$$M \cdot \vec{u} = \vec{a}$$

$$u = M^{-1} \cdot \vec{a}$$

$$y_p = L^{-1} f \qquad y = y_p + y_H$$

$$M^{-1} \cdot M = I$$

$$\sum (M^{-1})_{ik} M_{kj} = \delta_{ij}$$

$$(Ly)(x) = \int L(x, x') y(x') dx'$$

$$\int L^{-1}(x, x') L(x', x'') dx' = \delta(x - x')$$

$$= \int L(x,x') L^{-1}(x',x'')\, dx'$$

Green's function

$G(x,x')$

$$L(x,x'')\, G(x'',x')\, dx'' = \delta(x-x')$$

$$L\, G(x,x') = \delta(x-x')$$

$$\left[a_2(x) \frac{\partial^2}{\partial x^2} + a_1(x) \frac{\partial}{\partial x} + a_0(x)\right] G(x,x') = \delta(x-x')$$

$$y_p(x) = \int G(x,x')\, f(x')\, dx'$$

$$L_x F(x,x') = 0$$

$$G(x,x') + F(x,x')$$

In $\mathbb{R}^3$

$$L_{\vec{r}}\, G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$= \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$L\, y(\vec{r}) = f(\vec{r})$$

$$y_p(\vec{r}) = \int G(\vec{r},\vec{r}\,')\, f(\vec{r}\,')\, d^3\vec{r}$$

$$y(\vec{r}) = y_p(\vec{r}) + y_r(\vec{r})$$

<u>Electrostatics</u>

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\vec{\nabla}\times\vec{E} = \vec{0}$$

$$\vec{E} = -\vec{\nabla}\Phi$$

$$\nabla^2\Phi = -\rho/\varepsilon_0$$

$$\Phi_\rho(\vec{r}) = \int G(\vec{r},\vec{r}\,')\,\frac{\rho(\vec{r}\,')}{\varepsilon_0}\,d^3\vec{r}\,'$$

$$L_{\vec{r}}\,G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,') \qquad L_{\vec{r}}\,F(\vec{r}) = \delta^{(3)}(\vec{r})$$

Suppose L is translation invariant
so if $\vec{a}$ is a constant vector

$$L_{\vec{r}+\vec{a}} = L_{\vec{r}}$$

$$L_x = D^2 = \frac{d^2}{dx^2} \qquad \tilde{x} = x+a$$

$$\frac{d}{d\tilde{x}} = \frac{dx}{d\tilde{x}}\frac{d}{dx} = \frac{d}{dx}$$

$$\tilde{L} = \tilde{D}^2 = D^2$$

$$L_{\vec{r}+\vec{a}}\,G(\vec{r}+\vec{a},\vec{r}\,') = \delta^{(3)}(\vec{r}+\vec{a}-\vec{r}\,^2)$$

$$= \delta^{(3)}(\vec{r})$$

$$\boxed{L_{\vec{r}+\vec{r}} = L_{\vec{r}}} \qquad = L_{\vec{r}}\,G(\vec{r}+\vec{r}\,',\vec{r}\,')$$

$$= L_{\vec{r}}\,G(\vec{r},0)$$

$$G(\vec{r}+\vec{r},\vec{r}') = G(\vec{r},\vec{0})$$

$$G(\vec{r}-\vec{r}'+\vec{r}',\vec{r}') = G(\vec{r}-\vec{r}',\vec{0})$$

$$G(\vec{r},\vec{r}') = G(\vec{r}-\vec{r}',0)$$

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2_{\vec{r}} G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

$$G(\vec{r},\vec{r}') = F(\vec{r}-\vec{r}')$$

$$\nabla^2_{\vec{r}} F(\vec{r}) = \delta^{(3)}(\vec{r})$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) F(x,y,z) = \delta(x)\delta(y)\delta(z)$$

$$G(\vec{r},\vec{r}') = F(x-x', y-y', z-z')$$

$$F(\vec{r}) = F(r)$$

$$\nabla^2 F(\vec{r}) = \delta^{(3)}(\vec{r})$$

$$\nabla^2_{\vec{r}} F(\vec{r}) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right)$$

$$r > 0$$

$$\nabla^2_{\vec{r}} F(\vec{r}) = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right) = 0$$

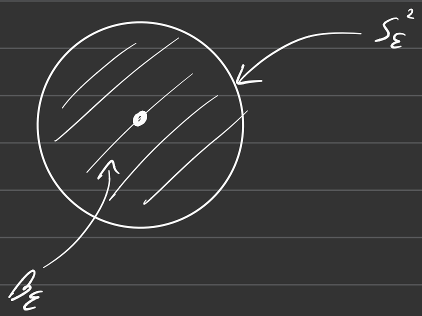
$$r^2\frac{dF}{dr} = C$$

$$\frac{dF}{dr} = \frac{C}{r^2}, \qquad F(r) = -\frac{C}{r} + A$$

$$\nabla^2_{\vec r} F(\vec r) = \delta^{(3)}(\vec r)$$

$$\int_{B_\varepsilon} \nabla^2_{\vec r} F(\vec r)\, d^3\vec r = 1 = \int_{B_\varepsilon} \vec\nabla\cdot(\vec\nabla F)\, d^3\vec r$$

$$= \oint_{S^2_\varepsilon} \vec\nabla F \cdot d\vec\sigma$$



$$= \oint_{S^2_\varepsilon} \frac{dF}{dr}\, \hat e_r \cdot \hat e_r\, d\sigma$$

$$= \left.\frac{dF}{dr}\right|_\varepsilon 4\pi\varepsilon^2$$

$$= \left(\frac{C}{\varepsilon^2}\right)(4\pi\varepsilon^2)$$

$$= 4\pi C$$

$$\Rightarrow C = \frac{1}{4\pi}$$

$$F(r) = -\frac{1}{4\pi}\frac{1}{r}$$

$$= F(|\vec r|)$$

$$G(\vec{r},\vec{r}') = F(\vec{r}-\vec{r}')$$

$$= -\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}$$

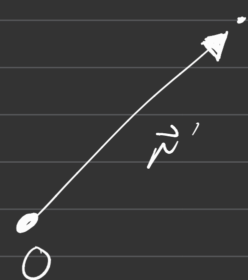
$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$$

$$\Phi_\rho(\vec{r}) = \int \left(-\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}\right)\left(-\frac{\rho(\vec{r}')}{\varepsilon_0}\right) d^3r$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r$$

$$\nabla^2_{\vec{r}}\, G(\vec{r},\vec{r}') = \delta^{(3)}(\vec{r}-\vec{r}')$$

$$\nabla^2_{\vec{r}}\left[-\frac{q}{\varepsilon_0} G(\vec{r},\vec{r}')\right] = -\frac{q\,\delta^{(3)}(\vec{r}-\vec{r}')}{\varepsilon_0}$$



$$\boxed{\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')}$$

$$\nabla^2_{\vec{r}}\, G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\nabla^2_{\vec{r}}\, F(\vec{r}) = \delta^{(3)}(\vec{r})$$

Fourier Transforms

$$F(\vec{r}) = \int \frac{dk_x\, dk_y\, dk_z}{(2\pi)^{\frac{3}{2}}}\, e^{i(k_x x + k_y y + k_z z)}\, \tilde{F}(k_x, k_y, k_z)$$

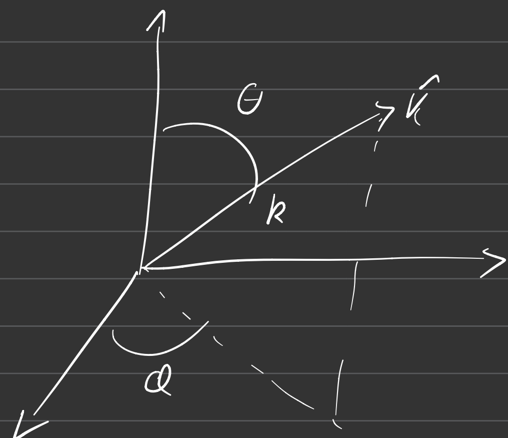
$$= \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}}\, e^{i\vec{k}\cdot\vec{r}}\, \tilde{F}(\vec{k})$$

$$\nabla^2 F = \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}} \left(-k_x^2 - k_y^2 - k_z^2\right) \tilde{F}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$$= -\int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}}\, |\vec{k}|^2\, \tilde{F}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$$\delta^{(3)}(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{|\vec{k}|^2 \tilde{F}(\vec{k})}{2\pi^{3/2}} = \frac{1}{(2\pi)^3}$$



$$\tilde{F}(\vec{k}) = -\frac{1}{(2\pi)^{3/2}} \frac{1}{|\vec{k}|^2}$$

$$\Rightarrow F(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{|\vec{k}|^2}$$

$$= -\int \frac{\cancel{k}^2 \sin\theta \, dk \, d\theta \, d\phi}{(2\pi)^3} \frac{e^{ikr\cos\theta}}{\cancel{k}^2}$$

$$F(\vec{r}) = -\frac{1}{(2\pi)^2} \int_0^\infty dk \underbrace{\left(\int_0^\pi e^{ikr\cos\theta} \sin\theta \, d\theta \right)}$$

$$-\frac{1}{ikr} e^{ikr\cos\theta} \Big|_0^\pi$$

$$= \frac{2i\sin(kr)}{ikr}$$

$$= -\frac{1}{(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{\sin(kr)}{k} \, dk$$

$$= -\frac{1}{4\pi^2 r}\underbrace{\int_{-\infty}^{\infty} \frac{\sin(kr)}{k}\, dk}_{\pi}$$

$$= -\frac{1}{4\pi r}$$

$$G(\vec{r}, \vec{r}\,') = F(\vec{r} - \vec{r}\,')$$

$$= \frac{-1}{4\pi} \frac{1}{|\vec{r} - \vec{r}\,'|}$$

$$\nabla^2 \Phi = \frac{-\rho}{\varepsilon_0}$$

$$\Downarrow$$

$$\underbrace{\left(\frac{-1}{c^2} \frac{\partial}{\partial t} + \nabla^2\right)} \Phi = \frac{-\rho}{\varepsilon_0}$$

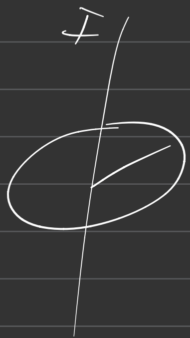
$$\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) G(t, \vec{r}, t', \vec{r}\,') = \delta(t - t')\, \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$\Rightarrow \Phi_\rho(t,\vec{r}) = -\int G(t,\vec{r},t',\vec{r}')\,\frac{\rho(t',\vec{r}')}{\varepsilon_0}\,d^3\vec{r}'\,dt'$$

$$G(t,\vec{r},t',\vec{r}') = -\frac{1}{4\pi|\vec{r}-\vec{r}'|}\,\underbrace{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}_{t' = t - \frac{|\vec{r}-\vec{r}'|}{c}}$$

Parallels Between electro and magnetostatics

Biot Savart Law

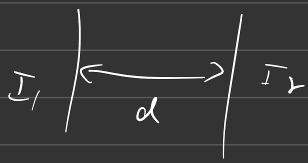


You can show from Biot-Savart law that Ampère's Law works

$$\vec{B}\, 2\pi s = I_{enc}$$

current enclosed by the Amperian loop

$$B = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}$$



$$F_{mag} = q(\vec{v} \times \vec{B})$$

$$= \frac{I_1 \mu_0 I_2}{2\pi d}$$

$$\vec{B}(\vec{r}) = \int \frac{\mu_0}{4\pi} \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Amper's Law in differential form

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

For a surface current $\vec{K}$

$$\frac{\partial \bar{A}}{\partial n}\text{above} - \frac{\partial \bar{A}}{\partial n}\text{below} = -\mu_0 \vec{K}$$

$$\hat{P} \cdot \hat{n} = \sigma_b$$

$$-\vec{\nabla} \cdot \vec{P} = \rho_b$$

Like electrostatics V has a multipole expansion. So does $\vec{A}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\text{ᴧ}}\,d\tau'$$

$$= -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J$$

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})} = \mu_0 \vec{J}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A}$$

$$= \mu_0 \vec{J}$$

$$-\nabla^2 \vec{A} = \mu_0 J$$

$$\bar{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\,d\tau'}{\text{ᴧ}}$$

Expand $\qquad \dfrac{1}{\mathscr{r}} = \dfrac{1}{r} \sum_n \left(\dfrac{r'}{r}\right)^n P_n(\cos\alpha)$

$A_{\text{monopole}} + A_{\text{dipole}} + \ldots$

$$A_{dip} = \frac{\mu_0 I}{r^2} \int \underbrace{(r'\cos\alpha)}_{=(r'\cdot\hat{r})} \overline{dl}'$$

$$\bar{m} = I \oint d\bar{a}$$

claims $\qquad \bar{A}_{dip} = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m}\times\hat{r}}{r^2}$

claim $\qquad \oint (\hat{r}\cdot\vec{r}')\,\overline{dl}' = \int \overrightarrow{da}'\times\hat{r}$

$$= -\hat{r}\times\int \overline{da}'$$

$T = (\vec{r}\cdot\hat{r})$

$\vec{\nabla}'(T) = \vec{\nabla}'(\hat{r}\cdot\vec{r}')$

$$= \hat{r} \times \cancelto{0}{(\vec{\nabla}' \times \vec{r}')}$$

$$+ (\hat{r} \cdot \vec{\nabla}')\vec{r}'$$

$$= (\vec{r} \cdot \vec{\nabla}')\vec{r}'$$

$$= \hat{r}$$

$$\oint T \, d\vec{l} = -\int (\vec{\nabla} T) \times d\vec{a}'$$

$$= -\int \hat{r} \times d\vec{a}'$$

$$= \int d\vec{a}' \times \hat{r}$$

$$\vec{v} = \vec{c}\, T$$

constant velocity

$$\vec{\nabla} \times \vec{c} = 0$$

$$\int (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$$

$$= \int (\vec{\nabla} \times \vec{c}\,T) \cdot d\vec{a}$$

$$= \int \Big[\underbrace{T(\vec{\nabla} \times \vec{c})}_{0} - \vec{c} \times (\vec{\nabla} T) \Big] \cdot d\vec{a}$$

$$= -\int \Big[\vec{c} \times (\vec{\nabla} T) \Big] \cdot d\vec{a}$$

$$\vec{c} \times \vec{\nabla} T \cdot d\vec{a} = \vec{c} \cdot (\nabla T \times d\vec{a})$$

$$B_{dip} = \vec{\nabla} \times A_{dip}$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \left(3(\vec{m} \cdot \vec{r})\hat{r} - \vec{m} \right)$$

$$A_{dip} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}$$

Friday 26th April 2 slots

May 3rd 2 slots

Let her know by 22nd April

$\vec{A}$ $\qquad \vec{\nabla} \times \vec{A} = \vec{B}$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}} \, d\tau'$$

Under the Assumption $\underbrace{\vec{\nabla} \cdot \vec{A} = 0}_{\text{gauge condition on freedom}}$

Exams Maxwell equations will be given

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$$

Similar relations exist for

$\vec{B}$, $\vec{H}$ $\qquad \vec{P} \longrightarrow \vec{M}$

$$
\begin{cases}
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \cdot \vec{B} = 0 \\
\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{cases}
\qquad
\begin{cases}
\vec{E} = -\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right) \\
\vec{B} = \vec{\nabla} \times \vec{A}
\end{cases}
$$

$$\nabla \cdot (\vec{\nabla} \times \vec{E}) = 0$$

$$\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B}) = 0$$

For electrostatics

$$\vec{E} = -\vec{\nabla} V$$

not ok for dynamics

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{\nabla} \cdot \vec{B} = 0$ even ok for dynamics

$$\vec{\nabla} \times \vec{E} = \frac{-\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right)$$

so $$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{A}}{\partial t} = 0 = \vec{\nabla} \times \left(-\vec{\nabla} V\right)$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

$$\vec{E} = -\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right)$$

hint hint
remember this

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{-\partial}{\partial t}\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right)$$

$$= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \left(-\frac{\partial}{\partial t} \vec{\nabla} V - \frac{\partial^2 \vec{A}}{\partial t^2}\right)$$

$$\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \varepsilon_0 \vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2}$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J} - \nabla\left(\nabla \cdot A + \frac{1}{c^2}\frac{\partial V}{\partial t}\right)$$

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{c^2}\partial_t^2$$

$$= \frac{1}{c}\Box$$

Preferred gauge for ED is Lorentz gauge which

$$\frac{1}{c^2}\Box \vec{A} = -\mu_0 \vec{J}$$

$$\frac{1}{c^2}\Box = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$$

$$A' = \bar{A} + \bar{\alpha}$$

$$V' = V + B$$

$$\vec{\nabla} \times \vec{A} = \vec{B} = \vec{\nabla} \times \vec{A}$$

$$= \vec{\nabla} \times (\vec{A} + \vec{\alpha})$$

$$= (\vec{\nabla} \times \vec{A}) + (\vec{\nabla} \times \vec{\alpha})$$

$$\vec{\alpha} = \vec{\nabla} \lambda$$

$$\vec{E} = -\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\vec{\nabla}(V + \beta) - \left(\frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{\alpha}}{\partial t}\right)$$

So $$\vec{\nabla} \beta + \frac{\partial \vec{\alpha}}{\partial t} = 0$$

$$= \vec{\nabla} \beta + \frac{\partial \nabla \lambda}{\partial t}$$

$$= \vec{\nabla} \underbrace{\left(\beta + \frac{\partial \lambda}{\partial t}\right)} = 0$$

$$\beta + \frac{\partial \lambda}{\partial t} = u(t)$$

$$\lambda' = \lambda - \int_0^t u(t')\,dt'$$

maximal possible changes

$$V(\vec{r}, t) = 0 \longrightarrow \nabla V = 0, \; \frac{\partial V}{\partial t} = 0$$

$$\vec{A}(\vec{r}, t) = \frac{-qt}{4\pi\varepsilon_0 r^2} \hat{r}$$

Find $\vec{E}, \vec{B}$ determine how the configuration is for charge and currents

$$\vec{B} = \vec{\nabla} \times \vec{A} = 0$$

$$\vec{E} = -\left(\nabla V + \frac{\partial \vec{A}}{\partial t}\right)$$

$$= 0 + \frac{q\hat{r}}{4\pi\varepsilon_0 r^2}$$

$$\vec{E} = \frac{q\hat{r}}{4\pi\varepsilon_0 r^2}$$

= That of a point charge q located at origin and stationary

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\nabla\cdot\left(\frac{\partial\vec{A}}{\partial t} + \nabla V\right) = -\rho/\varepsilon_0$$

or $$\nabla^2 V + \nabla\cdot\frac{\partial\vec{A}}{\partial t} = -\rho/\varepsilon_0$$

$$\vec{\nabla}\cdot\vec{A} = \mu_0\varepsilon_0\frac{\partial V}{\partial t}$$

$$\nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon$$

TE TM TEM

Waves (Q6) in Vacuum

$$\nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0 \quad = 0$$

$$\nabla^2\vec{A} = \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J} \quad = 0$$

$$\Box V = \Box\vec{A} = 0$$

Magnetic Fields in Matter

P $\quad M$ magnetic dipole moment per unit volume

D $\quad H$

E $\quad B$ $\quad \vec{H} \sim \vec{B}, \vec{M}$

ρ_b $\quad J_b \rightarrow$ vol

σ_b $\quad K_b \rightarrow$

ρ_f $\quad J_f$

σ_f $\quad K_f$

As we did in last class

$$J_b = \vec{\nabla} \times \vec{M}$$

so $\vec{\nabla} \cdot J_b = 0$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_b$$

$$\vec{J} = \vec{J}_f + \vec{J}_b$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_f + \vec{J}_b)$$

$$= \mu_0 J_f + \mu_0 (\vec{\nabla} \times \vec{M})$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \boxed{J_f = \vec{\nabla} \times \vec{H}}$$

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}}$$

hint hint

$$\vec{\nabla} \times \vec{H} = J_f$$

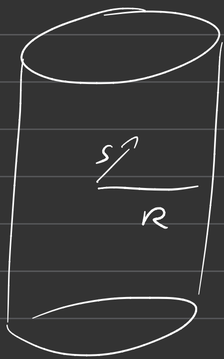
$$\oint \vec{H} \cdot dl = I_{free}$$

$$\vec{D} = \varepsilon E = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$= \frac{\vec{B}}{\mu}$$ for linear magnetic materials

$\vec{H}$ inside and outside for this wire, suppose I want $s < R$



$$\vec{H} \cdot 2\pi s \, \hat{\phi} = \frac{I \pi s^2}{\pi R^2}$$

$$\vec{H} = \frac{I s}{2R^2} \hat{\phi} \qquad s > R$$

If $s > R$

$$\vec{H} \cdot 2\pi s = I \qquad \Bigg| \qquad \vec{B} = \mu_0 \vec{H}$$

$$\vec{H} = \frac{I}{2\pi s} \qquad \Bigg| \qquad = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

Boundary Conditions

wrt K_f, I_f, J_f change

$$B^{\perp}_{above} - B^{\perp} = 0$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 (\hat{K} \times \hat{n})$$

In presence of matter

$$H^{\parallel}_{above} - H^{\parallel}_{below} = \hat{K}_f \times \hat{n}$$

Example

$$V = 0 \, , \quad \vec{A} = \frac{\mu_0 K}{4c}(ct - |x|)^2 \hat{z} \qquad |x| < ct$$

$$= 0 \quad \text{otherwise}$$

Discontinuity exists

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 K}{2}(ct - |x|)\hat{z}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \frac{\mu_0 K}{4c}(ct - |x|)^2 \end{vmatrix}$$

ha works tho

$$= -\hat{y}\, \partial_x \left(\frac{\mu_0 K}{4c}(ct - |x|)^2 \right)$$

$$\vec{B} = \pm \frac{\mu_0 K}{2c}(ct - |x|)$$

Positive for $x > 0$
and negative for $x < 0$

$\vec{K}$ can be computed

$$\hat{K} \times \hat{x} = \hat{y}$$

$$\hat{K} = Kt\hat{z}$$

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} = 0$$

$$\vec{E} = \frac{\mu_0 K}{2} (ct - |x|)\hat{z}$$

$$\vec{\nabla} \times \vec{E} = \mu_0 \varepsilon_0 \frac{\partial B}{\partial t} + (J)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} + (J)$$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & \pm\frac{\mu_0 K}{2c}(ct - |x|) & 0 \end{vmatrix}$$

$$= \hat{z} \text{ term}$$

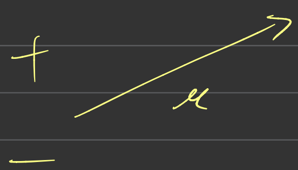
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \frac{\mu_0 K}{2}(ct - |x|) \end{vmatrix}$$

$= \text{along } \hat{y}$

Exam hint

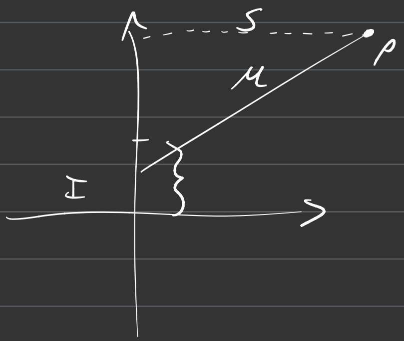
Question

on dipole radiation something



Retarded and Advanced potentials

Only retarded potentials make sense physically



$(V, \vec{A})$

Electric potential

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r', t)}{\mathscr{r}} \, dz'$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I(z')}{\mathscr{r}} \, dz'$$

Time when you record $V, \vec{A}$ is different than that current / charge producing $\vec{A}, V$

$$t_{retarded} = t_r = t - \frac{\mathscr{r}}{c}$$

Similarly $t_{advanced} = t_a = t + \frac{r}{c}$

also solves Maxwells equations etc

$$\Box^2 \vec{A} = -\mu_0 \vec{J}$$

$$\Box^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2$$

Question here

$$\vec{\nabla} V = ?$$

$$\vec{\nabla} \times \vec{A} = ?$$

$\vec{E}$ and $\vec{B}$ given by Jeffimenko equations

Let try to see if

$$\Box^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} V = \frac{1}{4\pi\varepsilon_0} \int \left[\nabla \rho \, \frac{1}{\mathscr{r}} + \rho \nabla\left(\frac{1}{\mathscr{r}}\right)\right] d\tau'$$

$$\boxed{\frac{\partial}{\partial t_r} = \frac{\partial}{\partial t}}$$

$$\nabla t_r = -\frac{1}{c} \nabla \mathscr{r} = -\frac{\hat{\mathscr{r}}}{c}$$

$$t_r = t - \frac{\mathscr{r}}{c}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = \frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\nabla \rho(r', t_r) = \frac{\partial \rho}{\partial t_r} \nabla(t_r)$$

$$= \frac{\partial \rho}{\partial t} \nabla(t_r)$$

$$= -\dot{\rho} \frac{\hat{\mathscr{r}}}{c}$$

$$\nabla V = \frac{1}{4\pi\varepsilon_0} \int \left(-\frac{\dot{\rho}}{c} \frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$$\nabla^2 V = \frac{-1}{4\pi\varepsilon_0}\int\left[\left(\frac{1}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}}\nabla\dot{\rho} + \frac{\dot{\rho}}{c}\nabla\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right.\right.$$

$$\left.+\left((\nabla\rho)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \rho\vec{\nabla}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)\right]d\tau'$$

$$= \frac{-1}{4\pi\varepsilon_0}\int\frac{\hat{\mathscr{r}}}{c\mathscr{r}}\nabla\dot{\rho} + \rho\vec{\nabla}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\Big)\,d\tau'$$

$$\nabla\dot{\rho} = -\ddot{\rho}\frac{\hat{\mathscr{r}}}{c}$$

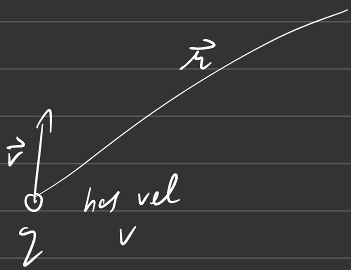
$$\nabla\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta(\vec{\mathscr{r}})$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int\frac{\ddot{\rho}}{c^2\mathscr{r}} - \frac{1}{c}\int\delta^3(\vec{\mathscr{r}})\rho$$

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}V - \frac{\rho(r,t)}{\varepsilon_0}$$

$$\Box^2 V =$$

Arbitrary Motion of a charge q

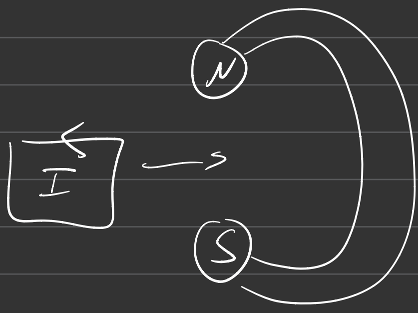


$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(rc - \hat{r}\cdot\vec{v})}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

Lienard Wichert potentials

Faraday



Experiment (1)

Force on the loop
no Lorentz force

→ Moving charge
so Lorentz force

Experiment (2)

Moved the magnetic
field and found same
I, Emf

→ moving magnet

$$\Rightarrow \frac{\partial B}{\partial t} \sim E$$

So electromagnetic theory inter-related
wrt motion of charges and that
of magnetic fields and if such a
thing can be incorporated to SR

Reminder of special relativity

Lorentz transformations

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

S is at rest

$$\beta = \frac{v}{c}$$

S' is moving at velocity

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$\vec{A} \cdot \vec{B}$

A is covariant

(a^0, a^1, a^2, a^3)

Covariant A

$(-a^0, a^1, a^2, a^3)$

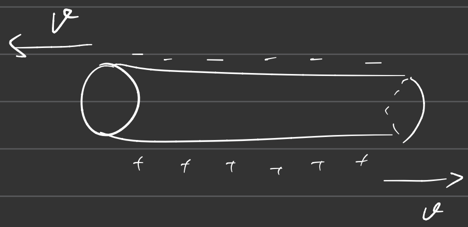
$$\eta^{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$p^\mu = \eta^\mu$ when $t = \tau$

$$\eta^\mu = m\frac{du^\mu}{d\tau}$$

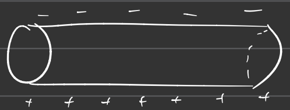
$$F = \frac{dp}{dt}$$

Magnetostatics from Electrostatics



Frame S

$\longrightarrow x$



$|\lambda|$ = charge density for positive and negative

$q \; \overset{\circ\longrightarrow}{u}$ $\quad S'$ $\qquad$ Current $= I = 2\lambda v$

$u < v$

u is also along ~~the~~ x axis

Charge q experiences an electric force due to length contraction

$$v_{\pm} = \frac{v \mp u}{1 \mp \frac{vu}{c^2}}$$

(using Einstein's velocity addition formula)

Given that λ_0 is the charge density seen by the charges

$$\lambda = \gamma \lambda_0 \qquad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

I can also write

$$\gamma_\pm = \frac{1}{\sqrt{1 - \frac{v_\pm^2}{c^2}}}$$

So similarly $\lambda_\pm$ can be defined and

$$\lambda_\pm = \pm \gamma_\pm \lambda_0 \longrightarrow \text{is the charge density observed by } \bar{S} \text{ or } S \text{ frame}$$

$$= \pm \gamma_\pm \frac{\lambda}{\gamma}$$

Now lets calculate

$$\gamma_\pm = \frac{1}{\sqrt{1 - \frac{1}{c^2}\frac{(v \mp u)^2}{\left(1 \pm \frac{vu}{c^2}\right)^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{c^2 (v \mp u)^2}{(c^2 \pm vu)^2}}}$$

$$= \frac{(c^2 \mp vu)}{\sqrt{(c^2 \mp vu)^2 - c^2(v \mp u)^2}}$$

$$= \frac{(c^2 \mp uv)}{\sqrt{c^4 \mp 2vuc^2 + v^2u^2 - c^2v^2 - c^2u^2 \pm 2uvc^2}}$$

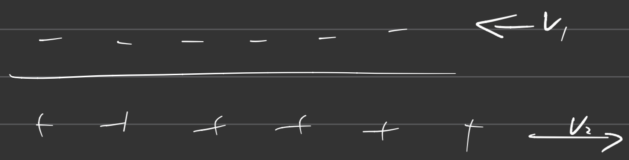
$$= \frac{c^2 \mp uv}{\sqrt{c^4 - c^2v^2 - c^2u^2 + u^2v^2}}$$

$$= \frac{c^2 \mp uv}{\sqrt{(c^2 - u^2)(c^2 - v^2)}}$$

$$= \frac{1 \mp \frac{uv}{c^2}}{\sqrt{(1 - \frac{u^2}{c^2})(1 - \frac{v^2}{c^2})}} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{\gamma(1 \mp \frac{uv}{c^2})}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\lambda_{total} = \lambda_0(\gamma_+ - \gamma_-) = \lambda_+ + \lambda_-$$



(S) frame

$v - u > 0$

$v + u > 0$

$\downarrow$

$\vec{E} = 0$ due to line charges

$q \xrightarrow[u > v]{}$

$$v_\pm = \frac{u - v}{1 - \frac{vu}{c^2}}$$

$$\gamma_\pm = \frac{\gamma\left(1 \mp \frac{uv}{c^2}\right)}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\lambda_{total} = \lambda_0 (\gamma_+ - \gamma_-)$$

$$= \frac{\lambda_0 \gamma}{\sqrt{1 - \frac{u^2}{c^2}}} \left(1 - \frac{uv}{c^2} - 1 - \frac{uv}{c^2}\right)$$

$$= \frac{\lambda_0 \gamma}{\sqrt{1-\frac{u^2}{c^2}}} \left(\frac{-2uv}{c^2} \right)$$

$$= \frac{-2uv\lambda}{c^2\sqrt{1-\frac{u^2}{c^2}}} \qquad \lambda = \lambda_0 \gamma$$

Current in S frame $= 2\lambda v$

$$E = \frac{\lambda_{total}}{2\pi\varepsilon_0 s}$$

$$qE = F_{elec} = \frac{\lambda_{total}\, q}{2\pi\varepsilon_0 s}$$

$$= \frac{-2uv\lambda q}{c^2\sqrt{1+\frac{u^2}{c^2}}\,(2\pi\varepsilon_0 s)}$$

$$= \frac{-2uv\lambda q\,\mu_0}{\sqrt{1+\frac{u^2}{c^2}}\; 2\pi s}$$

This IS Gonna be on ~~the~~ Exam

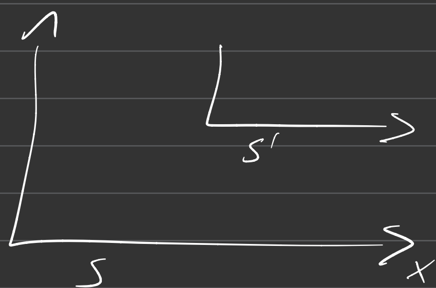
$$B = \frac{\mu_0 I}{2\pi s} \quad , \quad qE = \frac{-uB}{\sqrt{1-\frac{u^2}{c^2}}}$$

Go back ~~to~~ S frame

$$F = \sqrt{1-\frac{u^2}{c^2}}\, F'$$

Transformations of $\vec{E}$ and $\vec{B}$

S' are moving with velocity v along x axis



$$E_x' = E_x$$

$$E_x' = \gamma(E_y - vB_z)$$

$$E_z' = \gamma(E_z + vB_y)$$

$$B_x' = B_x$$

$$B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Good to remember

$E \cdot B$

$E^2 - cB^2$ are invariant

$\underline{E \cdot B \text{ is invariant}}$

$E_x B_x = E_x B_x'$

$$E_y' B_y' = \gamma^2 (E_y - v B_z)(B_y + \tfrac{v}{c^2} E_z)$$

$$= \gamma^2 (E_y B_y - \tfrac{v^2}{c^2} B_z E_z + \cancel{E_y E_z \tfrac{v}{c^2}} - \cancel{v B_y B_z})$$

$$E_z' B_z' = \gamma^2 (E_z + v B_y)(B_z - \tfrac{v}{c^2} E_y)$$

$$= \gamma^2 (E_z B_z - \tfrac{v^2}{c^2} B_y E_y - \cancel{\tfrac{v}{c^2} E_z E_y} + \cancel{v B_y B_z})$$

Add

$$E_x B_x + \cancel{\gamma^2} (E_y B_y)\cancel{(1 - \tfrac{v^2}{c^2})} + \cancel{\gamma^2} E_z B_z \cancel{(1 - \tfrac{v^2}{c^2})}$$

$$= E \cdot B$$

Prove $E^2 - c^2B^2$ is invariant

$$E_y'^2 = \gamma^2\left(E_y^2 - \cancel{2vE_yB_z} + v^2B_z^2\right)$$

$$E_z'^2 = \gamma^2\left(E_z^2 + \cancel{2vE_zB_y} + v^2B_y^2\right)$$

$$c^2B_y'^2 = \gamma^2\left(c^2B_y^2 + \frac{v^2}{c^2}E_z^2 + \cancel{2B_yE_zv}\right)$$

$$c^2B_z'^2 = \gamma^2\left(c^2B_z^2 + \frac{v^2}{c^2}E_y^2 - \cancel{2B_zE_yv}\right)$$

Add

$$\cancel{\gamma^2}\left(E_y^2+E_z^2\right)\cancel{\left(1-\frac{v^2}{c^2}\right)} - c^2\cancel{\gamma^2}\left(B_z^2+B_y^2\right)\cancel{\left(1-\frac{v^2}{c^2}\right)}$$

$$= E_y^2+E_z^2 - c^2\left(B_y^2+B_z^2\right)$$

E, B are components of 4×4 antisymmetric matrix (tensor)

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Poynting's Theorem

$$S = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$\downarrow$

This is the main ingredient for for dipole radiation of any radiation

$$F = dq(\vec{E} + \vec{v} \times \vec{B})$$

$$F \cdot dl = dW = dq\,\vec{E} \cdot \vec{v}\,dt$$

$$dl = v\,dt \qquad dq\,\vec{v}\,dt = \vec{J}\,dt$$

$$\frac{dW}{dt} = \int (\vec{E} \cdot \vec{J})\, d\tau$$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \vec{E} \frac{\partial \vec{E}}{\partial t}$$

$$\int (\vec{E} \cdot \vec{J}) \, d\tau = \int \vec{E} \cdot \left(\frac{\vec{\nabla} \times \vec{J}}{\mu_0} \right) - \varepsilon_0 \vec{E} \frac{\partial \vec{E}}{\partial t}$$

Poynting Theorem

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$dW = \vec{F} \cdot \vec{dl} = q\vec{E} \cdot \vec{v} \, dt$$

$q = \rho \, d\tau$ ← volume element

This gives

$$\frac{dW}{dt} = \int (\vec{E} \cdot \vec{J}) \, d\tau$$

as $\vec{J} \cdot d\tau = q\vec{v}$

$\vec{E} \cdot \vec{J}$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Now, $\vec{\nabla} \cdot (\vec{E} \times \vec{B})$

$$= \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\underbrace{\vec{\nabla} \times \vec{B}})$$

$$\vec{E} \cdot \vec{J} = \frac{\vec{E}}{\mu_0} \cdot \underbrace{\vec{\nabla} \times \vec{B}} - \vec{E} \cdot \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \left(\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right)$$

Now $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$= \frac{-1}{\mu_0} \vec{B} \frac{\partial \vec{B}}{\partial t} - \frac{1}{\mu_0} \vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

Now $\frac{\partial}{\partial t} (E^2) = 2\vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$

$$\vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

$$- \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$\frac{dW}{dt} = \int \vec{E}\cdot\vec{J}$$

$$= \underbrace{-\frac{d}{dt}\int \frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{B^2}{\mu_0}\right) d\tau}_{\text{Energy stored in the configuration}} \underbrace{- \frac{1}{\mu_0}\oint (\vec{E}\times\vec{B})\, da}_{\text{radiated energy}}$$

Laplace's Equation in Cylindrical co-ordinates

$$\nabla^2 V = 0$$

No dependence on z

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \cancel{\frac{\partial^2 V}{\partial z^2}}^{\,0} = 0$$

$$V(s,\phi) = R(s)\Theta(\phi)$$

$$\frac{s}{\Theta}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\phi^2} = k^2$$

$$\Theta(\phi + 2\pi) = \Theta(\phi)$$

(Not exp growth wrt ϕ)

$$\Theta = a\sin k\phi + b\cos k\phi$$

$$\frac{s}{R}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) = k^2$$

Try $R = s^n$

$$\frac{s\,\partial R}{\partial s} = n s^{n}$$

$$\frac{s}{R}\; n^2 s^{n-1}$$

$n^2 = k^2$ $\qquad n = \pm k$

$$\sum_{k=1}^{\infty} s^k (a_k \sin k\phi + b_k \cos k\phi)$$
$$+ s^{-k} (a'_k \sin k\phi + b'_k \cos k\phi)$$

$k = 0$

$$\frac{\partial^2 \Theta}{\partial \phi^2} = 0 \quad \Rightarrow \Theta = A\phi + B$$

$$\Theta(\phi + 2\pi) = \Theta(\phi)$$

so $\qquad A = 0$

R part

$$\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) = 0$$

$$s \frac{\partial R}{\partial s} = \lambda$$

$$\frac{\partial R}{\partial s} = \frac{\lambda}{s}$$

$$R = \lambda \ln s + c$$

$$V = R\Theta$$

Gen solution

$$a_0 \ln s + b_0$$
$$+ \sum_{k=1}^{\infty} (a_k \sin k\phi + b_k \cos k\phi) s^k$$
$$+ \sum_{k=1}^{\infty} (a_k' \sin k\phi + b_k' \cos k\phi) s^{-k}$$

If we do have z dependence

$$V(s,\phi,z) = R(s)\Theta(\phi)Y(z)$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{sR}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) + \frac{1}{\Theta s^2}\frac{\partial^2\Theta}{\partial \varphi^2} = \frac{-1}{Y}\frac{\partial^2 Y}{\partial z^2} = \lambda$$

$$\frac{s}{R}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) + \frac{1}{\Theta}\frac{\partial \Theta}{\partial \varphi} - \lambda s^2 = 0$$

$$\frac{s}{R}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) - \lambda s^2 = \frac{-1}{s}\frac{\partial^2\Theta}{\partial \varphi^2} = k_1^2$$

$$\frac{s}{R}\frac{\partial}{\partial s}\left(s\frac{dR}{ds}\right) - \lambda s^2 - k^2 = 0$$

$$\frac{1}{R}\frac{d^2R}{ds^2} + \frac{}{Rs}\frac{dR}{ds} - \lambda \frac{-k^2}{s^2} = 0$$

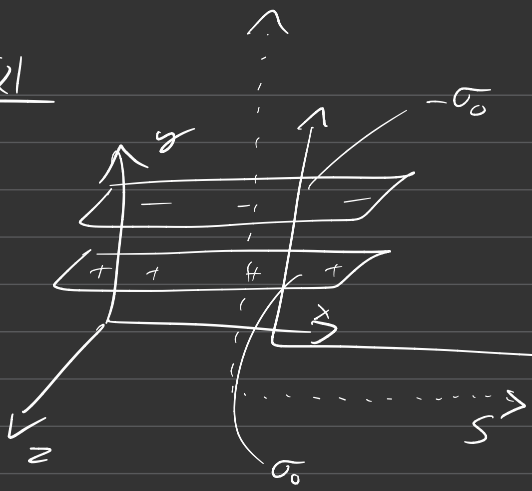
$\lambda \neq 0$, $\lambda = -\gamma^2$

$$\frac{1}{R}\frac{d^2R}{ds^2} + \frac{1}{Rs}\frac{dR}{ds} + \gamma^2 - \frac{k^2}{s} = 0$$

$s = r\gamma$ This gives bessel functions

Q1

$\mathcal{Z}$ 556



Parallel plate capacitors

$$E = \frac{\sigma_0}{\varepsilon_0}\hat{y}$$

Stationary is S_0 frame

S moves $v = v_0\hat{x}$

S observes with "Capacitance"

$$vel = -v_0\hat{x}$$

Cap: l_0 in x

(S_0) w in z

Q what is σ in S

$$\sigma_0 = \frac{Q}{l_0 w}$$

$$\sigma = \frac{Q}{l w}$$

$$l = \frac{l_0}{\gamma_0}$$

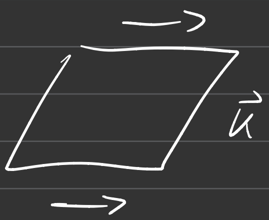
$$\boxed{\sigma = \gamma_0 \sigma_0}$$

$$E = \frac{\sigma}{\varepsilon_0} = \frac{\gamma_0 \sigma_0}{\varepsilon_0}$$

Charge density σ generates a current density on both plates

$K_+ = -\sigma v_0 \hat{x}$

$K_- = \sigma v_0 \hat{x}$



$B^{\parallel}_{abv} - B^{\parallel}_{bel} = \vec{K} \times \hat{n}$

K_+ along negative x

$$\frac{\mu_0}{2}(K_+ \times \hat{n}) = (-\sigma v_0 \hat{x} \times (-\hat{y}))\frac{\mu_0}{2}$$

$$= \frac{\mu_0}{2}\sigma v_0 \hat{z}$$

$$\frac{\mu_0}{2}(K_- \times \hat{n}) = \frac{\mu_0 \sigma v_0}{2}\hat{z}$$

$B_z = \mu_0 \sigma v_0$

$\bar{E}, \bar{B}$ are fields in $\bar{S}$

$$E_y = \frac{\sigma}{\varepsilon_0} \quad , \quad B_z = -\mu_0 \sigma v_0$$

$$\bar{v} = \frac{v_0 + v}{1 + \frac{v v_0}{c}} = \text{Rel vel}$$

$$\bar{\gamma} = \frac{1}{\sqrt{1 - \frac{\bar{v}^2}{c^2}}} \qquad \bar{\sigma} = \bar{\gamma}\sigma_0$$

$$S_0 : E_{0y} = \frac{\sigma_0}{\varepsilon_0} \quad , \quad \sigma = \sigma_0 \gamma_0$$

$$\bar{B}_z = -\frac{\bar{\gamma}_0}{\gamma_0} \mu_0 (\sigma \bar{v})$$

$$\bar{E}_y = \frac{\bar{\gamma}}{\gamma_0} \left(\frac{\sigma}{\varepsilon_0} \right)$$

$$\frac{\bar{\gamma}}{\gamma_0} = \gamma \left(1 + \frac{v v_0}{c^2} \right)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\bar{E}_y = \gamma\left(1 + \frac{vv_0}{c^2}\right)\frac{\sigma}{\varepsilon_0}$$

$$c^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$\bar{E}_y = \gamma\left(1 + vv_0\mu_0\varepsilon_0\right)\frac{\sigma}{\varepsilon_0}$$

$$= \gamma\left(\frac{\sigma}{\varepsilon_0}\right) + \gamma\mu_0 v v_0 \sigma$$

$$= \gamma E_y - \gamma v B_z$$

$$= \gamma\left(E_y - vB_z\right)$$

$$\bar{B}_z = -\frac{\bar{\gamma}}{\gamma_0}\mu_0\sigma v$$

$$= -\gamma\left(1 + \frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v + v_0}{1 + \frac{vv_0}{c^2}}\right)$$

$$= -\gamma\mu_0\sigma\left(v + v_0\right)$$

$$= -\gamma\mu_0\sigma v_0 - \underbrace{\gamma\mu_0\sigma v}_{B_z}$$

$$= \gamma B_z - \frac{\gamma \bar{E}_y v}{c^2}$$

Intro QFT

Mechanics

Classical → pt particle

Quantum → Wavefunctions

Energy

Lagrangians

→ Field Theories

fields

Operators

energy density

$\mathcal{L}$ (Density) / L

$S = \int L\,dt \longrightarrow \delta S = 0$

$S_{ft} = \int \mathcal{L}\, d^4x$

or $\int \mathcal{L} \sqrt{-g}\, d^4x$

Note $\det(g_{\mu\nu}) = \sqrt{-g}$

<u>Simple Systems</u>

<u>EOM</u>

m

$F \propto kx$

Infinite set of beads

conn by k = spring constant

mass = m for each bead

spacing a

$$V = \sum_{l} K(\eta_{i+1} - \eta_i)^2 \qquad F \propto Kx$$

$$T = \sum \frac{1}{2} m \dot{\eta}_i^2$$

$$L = T - V$$

$$= \frac{1}{2} \sum m \dot{\eta}_i^2 - \frac{1}{2} \sum a^2 K \left(\frac{\eta_{i+1} - \eta_i}{a} \right)^2$$

$$m = \mu a$$

$$L = \frac{1}{2} \sum \mu dx \dot{\eta}_i^2 - \frac{1}{2} \sum (Ka) dx \left(\frac{d\eta}{dx} \right)^2$$

$$= \frac{1}{2} \int \left[\mu \dot{\eta}_i^2 - y \left(\frac{d\eta}{dx} \right)^2 \right] dx$$

$$\mathcal{L} = \frac{1}{2} (\mu \dot{\eta}^2 - y \eta'^2)$$

In general

$$\mathcal{L} = \mathcal{L}(\eta_\rho, \partial_\mu \eta_\rho, x, t)$$

We can use variational principle and get ~~the~~ EOM

$$\delta S = \delta\left(\int \mathcal{L}\, d^4x\right) = 0$$

$$\eta_\rho = \eta_{\rho,0} + \alpha \zeta_\alpha$$

A ~~the~~

$$t_{2,\alpha} = 0$$

$\alpha = 0$ at Boundaries B

$\frac{\partial \eta_\rho}{\partial x}$ at $B = 0$

$\frac{\partial S}{\partial \alpha} = 0$ this is EOM

Euler Lagrange Eqn

$$\frac{\partial \mathcal{L}}{\partial \eta_\rho} - \partial_\mu \frac{\partial \mathcal{L}}{(\partial \eta_{\rho,\mu})} = 0$$

$$\boxed{\eta_{\rho,\mu} = \frac{\partial \eta_\rho}{\partial x^\mu}}$$

A ~~theory~~ can have various ~~terms~~ in $\mathcal{L}$

All ~~terms~~ in $\mathcal{L}$ must be

(1) Lorentz invariant

(2) Two derivatives max

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$A^\mu = (V, \vec{A}) \qquad c = \mu_0 = \varepsilon_0 = 1$$

For E/M A^μ

only have 2 derivatives max

$J^\mu A_\mu$ $\quad$ $F^{\mu\nu}F_{\mu\nu}$

$A_\mu A^\mu$ $\quad$ $J^\mu A_\mu$

$A^\mu A_\mu$

$$J^\mu \partial_\nu F^{\mu\nu}$$

Maxwells

$$\partial_\mu F^{\mu\nu} = J^\nu$$

or 0 iff $J^\mu = 0$

$$J^\mu = (\rho, \vec{J})$$

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\mathcal{L} \cong \# F^{\mu\nu} F_{\mu\nu} + \# J^\mu A_\mu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A$$

$$-\partial_\mu \, \partial \, F^{\rho\sigma} F_{\rho\sigma}$$

$$A^\mu = (cV, \bar{A})$$

$$\partial_\mu A^\mu = \frac{\partial(cV)}{\partial(ct)} + \bar{\nabla}\cdot\bar{A}$$

$$= \frac{\partial V}{\partial t} + \bar{\nabla}\cdot\bar{A} = 0$$

$\rightarrow$ Lorenz Gauge

$\partial_\mu F_{\mu\nu} = J^\nu$ from LG

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu$$

$$= \Box A^\nu - \partial^\nu(\partial_\mu A^\mu)$$

$$\Box A^\nu = J^\nu$$

Statement (Ch 11 and latter section of ch 10)

$\bullet$ has an accelerated motion

q

Thus charges radiate

Larmor's formula

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

$P \sim q^2 a^2$ $\qquad$ a = acceleration